Multiproduct-Firm Oligopoly: An Aggregative Games Approach*

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Abstract

This paper introduces a new class of demand systems to study oligopolistic pricing games with multiproduct firms. We prove that this class of demand systems coincides with the set of demand systems that can be derived from discrete/continuous choice with iid type 1 extreme-value taste shocks. We also show that these demand systems are integrable with quasi-linear preferences. The pricing game is aggregative and payoff functions are uni-modal, although not necessarily quasi-concave. Firms' fitting-in and best-response functions can be entirely summarized by a uni-dimensional sufficient statistic, called the iota-markup. This allows us to show that, under fairly weak conditions, the pricing game has a Nash equilibrium. Under stronger conditions, this equilibrium is unique. We also provide an algorithm which exploits the aggregative nature of the game to compute the pricing equilibrium with multiproduct firms and CES demands. The algorithm always converges. As an application, we derive a number of results on the dynamic optimality of myopic merger policy under differentiated Bertrand competition.

1 Introduction

We introduce a new class of demand systems that nests the standard multinomial logit and CES demand systems. Using this demand system, we analyze an oligopoly pricing model

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with multiproduct firms. Exploiting the aggregative games structure of the model, we prove existence of equilibrium under fairly general conditions and provide sufficient conditions for uniqueness of equilibrium. We apply the model to static and dynamic merger analysis.

Analyzing the behavior of multiproduct firms in oligopolistic markets appears to be of a first-order importance. Multiproduct firms are endemic and play an important role in the economy. Even when defining products quite broadly at the XXX-digit level, multiproduct firms account for XXX percent of sales and XXX percent of exports (XXXSOURCE). Similarly, many markets are characterized by oligopolistic competition: Even at the 5-digit industry level, concentration ratios are fairly high (XXXEVIDENCE).

While there has been a lot of interest in multiproduct firms in the industrial organization and international trade literatures, researchers have generally shied away from dealing with the theoretical difficulties arising in oligopolistic models with multiproduct firms. The first source of difficulties is the high dimensionality of firms' strategy sets. The second source is that even with "well behaved" demand systems such as the multinomial logit demand system, firms' payoff functions are typically not quasi-concave when firms offer multiple products (Spady, 1984; Hanson and Martin, 1996). The third is that action sets are not bounded, and that it is often difficult to find natural upper bounds on prices.[1] The fourth is that payoff functions typically fail to be supermodular or log-supermodular.

In light of these technical difficulties, it is perhaps not surprising that the burgeoning literature on multiproduct firms in international trade has focused almost exclusively on models of monopolistic competition (Bernard, Redding and Schott, 2011; Dhingra, 2013; Mayer, Melitz and Ottaviano, 2014; Nocke and Yeaple, 2014).[2] In industrial organization, multiproducts firms are at the heart of the literature on bundling but the existing models are highly stylized.[3] Multiproduct firms feature very prominently in the empirical industrial organization literature on demand estimation where marginal costs are backed out under the *assumption* that the pricing equilibrium exists and that first-order conditions are sufficient (Berry, 1994; Berry, Levinsohn and Pakes, 1995; Nevo, 2001).[4]

In the first part of this paper, we introduce a new class of quasi-linear demand systems which nests standard CES and multinomial logit demand systems with heterogeneous quality and price-sensitivity parameters. We provide necessary and sufficient conditions for the

[1]For example, in the case of multinomial logit demand without an outside option, even a single-product firm's best-response price goes to infinity when the prices of rivals' offerings become large.

[2]An exception is Eckel and Neary (2010) who study (identical) multiproduct firms in a Cournot model with linear demand.

[3]Much of the bundling literature focuses on monopoly and/or two goods only (XXXREFERENCES).

[4]A theoretical foundation of these assumptions is missing so far, and the assumptions are likely to be violated in applications.

demand system to be integrable, i.e., to be derivable from a representative consumer choice problem. We show that, under the same conditions, the demand system is also derivable from a discrete-continuous choice problem with random utility.[5] As the demand system satisfies the Independence of Irrelevant Alternatives (IIA) axiom, the representative consumer's indirect (sub-)utility function can be written as as a function of a single-dimensional aggregator, which is given by the sum (over all products) of transforms of product prices. This property implies that the pricing game between multiproduct firms is aggregative; that is, each firm's profit can be written as a function of its own prices and the single-dimensional aggregator.

In the second part of the paper, we study a pricing game between multiproduct firms with arbitrary product portfolios and product-level heterogeneity in marginal costs and qualities (price-sensitivity parameters). The dimensionality of the problem is reduced, first, because the pricing game is aggregative, and, second, by showing that a firm's multidimensional pricing strategy can be fully summarized by a unidimensional sufficient statistic. This allows us to establish equilibrium existence under mild conditions, and equilibrium uniqueness under more stringent conditions. In case equilibrium is not unique, equilibria can be Pareto-ranked for the players (firms), with firms' ranking being the inverse of consumers' ranking of equilibria. The reduction in the dimensionality of relevant strategy sets not only helps proving existence and uniqueness but also computing equilibria efficiently, as we show. In the special cases of CES and multinomial logit demands, an additional aggregation property obtains: A firm's product portfolio with associated qualities and marginal costs can be fully summarized by a unidimensional sufficient statistic.

In the third part of the paper, we apply the pricing game to static and dynamic merger analysis. For the special cases of CES and multinomial logit demands, we extend Nocke and Whinston's (2010) result on the dynamic optimality of myopic merger approval policy to mergers between arbitrary multiproduct firms.

2 The Demand System: Quasi-Linear Integrability and Discrete-Continuous Choice

2.1 The Demand System

Definition 1. *A demand component is a triple* $\left(\mathcal{N}, (h_k)_{k\in\mathcal{N}}, (g_k)_{k\in\mathcal{N}}\right)$*, where* $\mathcal{N}$ *is a finite and non-empty set,* g_k, h_k *are functions from* $\mathbb{R}_{++}$ *to* $\mathbb{R}_{++}$ *for every* $k \in \mathcal{N}$*, and* h_k *is strictly*

[5]See Novshek and Sonnenschein (1979), Hanemann (1984), Dubin and McFadden (1984), Smith (2004) and Chan (2006) for references on discrete-continuous choice.

decreasing for every $k \in \mathcal{N}$.
If, in addition, g_k is $\mathcal{C}^1$ and h_k is $\mathcal{C}^2$ for every $k \in \mathcal{N}$, then $\left(\mathcal{N}, (h_k)_{k\in\mathcal{N}}, (g_k)_{k\in\mathcal{N}}\right)$ is called a differentiable demand component.

Definition 2. *The demand system associated with demand component $\left(\mathcal{N}, (h_k)_{k\in\mathcal{N}}, (g_k)_{k\in\mathcal{N}}\right)$ is defined as follows: for every $(p_j)_{j\in\mathcal{N}} \in \mathbb{R}^{\mathcal{N}}_{++}$, for every $k \in \mathcal{N}$,*

$$D_k\left(\mathcal{N}, (h_j)_{j\in\mathcal{N}}, (g_j)_{j\in\mathcal{N}}\right)\left((p_j)_{j\in\mathcal{N}}\right) = \frac{g_k(p_k)}{\sum_{j\in\mathcal{N}} h_j(p_j)}. \quad (1)$$

Product $k \in \mathcal{N}$ is a variety of the differentiated product we are interested in. The assumption that h_k is strictly decreasing for every $k \in \mathcal{N}$ means that varieties are strict substitutes. This class of demand systems nests multinomial logit (with $g_i(p_i) = h_i(p_i) = \exp\left(\frac{a_i - p_i}{\mu}\right)$, $\mu > 0$) and CES (with $h_i(p_i) = p_i^{1-\sigma}$, $g_i(p_i) = p_i^{-\sigma}$, $\sigma > 1$) demands as special cases.

Notice that demand is aggregative in the following sense: the demand for variety i depends only on price p_i and on aggregator $H = \sum_{j\in\mathcal{N}} h_j(p_j)$. That is, demand satisfies the Independence of Irrelevant Alternatives (IIA) axiom. As we will see later on, this property implies that the multiproduct-firm pricing game we will analyze is aggregative, in the sense that the profit of a firm depends on the vector of prices it sets and on the aggregator H. This will allow us to establish existence and uniqueness of a pricing equilibrium.

Before turning our attention to the pricing game, we start by imposing consistency requirements on our class of demand systems. We first derive necessary and sufficient conditions under which demand system (1) can be derived from quasi-linear utility maximization. We will then show that these conditions are also necessary and sufficient for this demand system to be derivable from discrete-continuous choice.

2.2 Integrability

We recall the following definition from Nocke and Schutz (2015):

Definition 3. *Let $n \geq 1$ and $D : \mathbb{R}^n_{++} \longrightarrow \mathbb{R}^n_+$. We say that D is quasi-linearly integrable if there exists a function $u : \mathbb{R}^n_+ \to \mathbb{R} \cup \{-\infty\}$ such that for every $(p_0, p, y) \in \mathbb{R}_{++} \times \mathbb{R}^n_{++} \times \mathbb{R}_+$ such that $p \cdot D\left(\frac{p}{p_0}\right) \leq y$, vector $\left(\frac{1}{p_0}\left(y - p \cdot D\left(\frac{p}{p_0}\right)\right), D\left(\frac{p}{p_0}\right)\right)$ is the unique solution of*

$$\max_{(q_0,q)} \{q_0 + u(q)\} \ \ s.t.\ \ p_0 q_0 + p \cdot q \leq y,\ \ q_0 \geq 0 \text{ and } q \geq 0.$$

When this is the case, we say that u (resp. $v : p \in \mathbb{R}^n_{++} \mapsto u(D(p))$) is a direct (resp. indirect) subutility function for demand system D.

In words, D is quasi-linearly integrable if it can be derived from quasi-linear utility maximization. We now adapt the definition of quasi-linear integrability to demand components:

Definition 4. *A demand component is quasi-linearly integrable if the demand system associated with it is quasi-linearly integrable.*

Let $\left(\mathcal{N}, (h_k)_{k\in\mathcal{N}}, (g_k)_{k\in\mathcal{N}}\right)$ be a differentiable demand component. For every $k \in \mathcal{N}$, we put $\gamma_k \equiv \frac{\left(h_k'\right)^2}{h_k''}$ (whenever this expression is well-defined).

Theorem 1. *Let $\left(\mathcal{N}, (h_k)_{k\in\mathcal{N}}, (g_k)_{k\in\mathcal{N}}\right)$ be a differentiable demand component. The following statements are equivalent:*

(i) $\left(\mathcal{N}, (h_k)_{k\in\mathcal{N}}, (g_k)_{k\in\mathcal{N}}\right)$ is quasi-linearly integrable.

(ii) There exists a strictly positive scalar α such that, for every $k \in \mathcal{N}$, $g_k = -\alpha h_k'$. Moreover, $h_k'' > 0$ for every $k \in \mathcal{N}$, and $\sum_{k\in\mathcal{N}} \gamma_k \leq \sum_{k\in\mathcal{N}} h_k$.

When this is the case, function $v(p) = \alpha \log\left(\sum_{j\in\mathcal{N}} h_j(p_j)\right) + \beta$ is an indirect subutility function for the associated demand system.

Proof. The proof relies on Nocke and Schutz (2015)'s Theorem 1, which states that a continuously differentiable demand system is quasi-linearly integrable if and only if its substitution matrix is symmetric and negative semi-definite at every price vector. In Appendix A.1, we show that symmetry of the substitution matrix is equivalent to the existence of an $\alpha > 0$ such that $g_k = -\alpha h_k'$ for every $k \in \mathcal{N}$, and that negative semi-definiteness is equivalent to $h_k'' > 0$ for every $k \in \mathcal{N}$ and $\sum_{k\in\mathcal{N}} \gamma_k \leq \sum_{k\in\mathcal{N}} h_k$. □

Notice that the representative consumer's indirect utility is aggregative as well, in that it only depends on the value of aggregator $H = \sum_{j\in\mathcal{N}} h_j(p_j)$.

In the remainder of the paper, we will only work with integrable demand components. We will also normalize α to 1, assume that h_k is $\mathcal{C}^3$ for all k, and drop the g_k's from the definition of a demand component, since these functions can be readily derived from the h_k's. We therefore redefine demand components as follows:

Definition 5. *A smooth and integrable demand component is a pair $\left(\mathcal{N}, (h_k)_{k\in\mathcal{N}}\right)$ such that $\mathcal{N}$ is a finite and non-empty set, for every $k \in \mathcal{N}$, h_k is a $\mathcal{C}^3$ function from $\mathbb{R}_{++}$ to $\mathbb{R}_{++}$, $h_k' < 0$ and $h_k'' > 0$, and $\sum_{k\in\mathcal{N}} \gamma_k \leq \sum_{k\in\mathcal{N}} h_k$.*

The demand system associated with smooth and integrable demand component $\left(\mathcal{N},(h_k)_{k\in\mathcal{N}}\right)$ *is:*

$$D_k\left(\mathcal{N},(h_j)_{j\in\mathcal{N}}\right)\left((p_j)_{j\in\mathcal{N}}\right)=\frac{-h_k'(p_k)}{\sum_{j\in\mathcal{N}}h_j(p_j)},\ \forall k\in\mathcal{N},\ \forall (p_j)_{j\in\mathcal{N}}\in\mathbb{R}_{++}^{\mathcal{N}}. \tag{2}$$

We denote the set of demand systems that can be generated by a smooth and integrable demand component by $\mathcal{D}$.

Definition 6. *We say that smooth and integrable demand components* $\left(\mathcal{N}^1,(h_k^1)_{k\in\mathcal{N}^1}\right)$ *and* $\left(\mathcal{N}^2,(h_k^2)_{k\in\mathcal{N}^2}\right)$ *are equivalent if there exists a bijection* $\sigma:\mathcal{N}^1\longrightarrow\mathcal{N}^2$ *such that for every* $k\in\mathcal{N}^1$ *and* $(p_j)_{j\in\mathcal{N}^1}$,

$$D_k\left(\mathcal{N}^1,(h_j^1)_{j\in\mathcal{N}^1}\right)\left((p_j)_{j\in\mathcal{N}^1}\right)=D_{\sigma(k)}\left(\mathcal{N}^2,(h_j^2)_{j\in\mathcal{N}^2}\right)\left((p_{\sigma^{-1}(j)})_{j\in\mathcal{N}^2}\right).$$

In words, two smooth and integrable demand components are equivalent if they induce the same demand system (up to a relabelling of products). The following lemma will be useful in the next subsection:

Lemma 1. *Let* $\left(\mathcal{N},(h_k)_{k\in\mathcal{N}}\right)$ *be a smooth and integrable demand component. There exists an equivalent smooth and integrable demand component* $\left(\mathcal{N},\left(\tilde{h}_k\right)_{k\in\mathcal{N}}\right)$ *such that* $\tilde{h}_k\geq\tilde{\gamma}_k$ *for every* $k\in\mathcal{N}$.

Proof. See Appendix A.2. □

2.3 Discrete-Continuous Choice

Definition 7. *A discrete-continuous choice model of consumer demand is a pair* $\left(\mathcal{N},(v_j)_{j\in\mathcal{N}}\right)$, *where* $\mathcal{N}$ *is a finite and non-empty set, and, for every* $j\in\mathcal{N}$, v_j *is a* $\mathcal{C}^3$ *function from* $\mathbb{R}_{++}$ *to* $\mathbb{R}$ *such that* $v_j'<0$ *and* $v_j''\geq 0$.

For every j, v_j is an indirect subutility function in a quasi-linear economy in which only variety j and the outside good are available. Conditions $v_j'<0$ and $v_j''\geq 0$ are necessary and sufficient for v_j to be an indirect subutility function (see Nocke and Schutz, 2015). Let y be the consumer's income. We normalize the price of the outside good to 1. The consumer makes discrete-continuous choices as follows. He first observes all varieties' prices $(p_j)_{j\in\mathcal{N}}$, and a vector of taste shocks $(\varepsilon_j)_{j\in\mathcal{N}}$. If he chooses variety $k\in\mathcal{N}$, then he consumes $q_k(p_k)=-v_k'(p_k)$ units of product k (by Roy's identity), uses the rest of his income to consume the outside good, and receives utility $y+v_k(p_k)+\varepsilon_k$. Therefore, the consumer chooses variety k only if

$$y + v_k(p_k) + \varepsilon_k \geq y + v_j(p_j) + \varepsilon_j \ \forall j \in \mathcal{N}.$$

We assume that the components of vector $(\varepsilon_j)_{j\in\mathcal{N}}$ are identically and independently drawn from a type-1 extreme value distribution. Therefore, by Holman and Marley's theorem, variety k is chosen with probability

$$\begin{aligned}\mathbb{P}_k(p) &= \Pr\left(v_k(p_k) + \varepsilon_k = \max_{j\in\mathcal{N}} \left(v_j(p_j) + \varepsilon_j\right)\right), \\ &= \frac{\mathrm{e}^{v_k(p_k)}}{\sum_{j\in\mathcal{N}} \mathrm{e}^{v_j(p_j)}}.\end{aligned}$$

It follows that the expected demand for product k is given by

$$\mathbb{P}_k(p) q_k(p_k) = \frac{\mathrm{e}^{v_k(p_k)}}{\sum_{j\in\mathcal{N}} \mathrm{e}^{v_j(p_j)}} \left(-v_k'(p_k)\right).$$

This motivates the following definition:

Definition 8. *The demand system generated by discrete-continuous choice model* $\left(\mathcal{N}, (v_j)_{j\in\mathcal{N}}\right)$ *is:*

$$\Delta_k \left(\mathcal{N}, (v_j)_{j\in\mathcal{N}}\right) \left((p_j)_{j\in\mathcal{N}}\right) = \frac{\mathrm{e}^{v_k(p_k)}}{\sum_{j\in\mathcal{N}} \mathrm{e}^{v_j(p_j)}} \left(-v_k'(p_k)\right), \ \forall k \in \mathcal{N}, \ \forall \left(p_j\right)_{j\in\mathcal{N}} \in \mathbb{R}_{++}^{\mathcal{N}}. \quad (3)$$

We denote the set of demand systems that can be generated by a discrete-continuous choice model by $\mathbb{D}$*.*

Upon inspection of equations (2) and (3), it seems clear that, given a smooth and integrable demand component $\left(\mathcal{N}, (h_j)_{j\in\mathcal{N}}\right)$, one can construct a discrete-continuous choice model $\left(\mathcal{N}, (\log(h_j))_{j\in\mathcal{N}}\right)$ such that

$$D\left(\mathcal{N}, (h_j)_{j\in\mathcal{N}}\right) = \Delta\left(\mathcal{N}, (\log(h_j))_{j\in\mathcal{N}}\right),$$

although we still need to check that $\left(\mathcal{N}, (\log(h_j))_{j\in\mathcal{N}}\right)$ is indeed a discrete-continuous choice model. Conversely, given a discrete-continuous choice model $\left(\mathcal{N}, (v_j)_{j\in\mathcal{N}}\right)$, one can construct a smooth and integrable demand component $\left(\mathcal{N}, \left(\mathrm{e}^{v_j}\right)_{j\in\mathcal{N}}\right)$ such that

$$\Delta\left(\mathcal{N}, \left(v_j\right)_{j\in\mathcal{N}}\right) = D\left(\mathcal{N}, \left(\mathrm{e}^{v_j}\right)_{j\in\mathcal{N}}\right),$$

although, again, we still need to check that $\left(\mathcal{N}, \left(\mathrm{e}^{v_j}\right)_{j\in\mathcal{N}}\right)$ is indeed a smooth and integrable demand component. We state this result formally below, and prove it in the appendix:

Theorem 2. *$\mathcal{D} = \mathbb{D}$: the set of demand systems that can be generated by smooth and integrable demand components and the set of demand systems that can be generated by discrete-continuous choice coincide.*

Proof. See Appendix A.3. □

3 The Pricing Game: Existence of an Equilibrium

3.1 Definitions

Definition 9. *A pricing structure is a tuple $\left(\mathcal{N}, \left(h_k\right)_{k\in\mathcal{N}}, \mathcal{F}\right)$, where:*

- *$\left(\mathcal{N}, \left(h_k\right)_{k\in\mathcal{N}}\right)$ is a smooth and integrable demand component such that $|\mathcal{N}| \geq 2$,*
- *$\mathcal{F}$, the set of firms, is a partition of $\mathcal{N}$ such that $|\mathcal{F}| \geq 2$.*

Definition 10. *A pricing game is a tuple $\left(\mathcal{N}, \left(h_k\right)_{k\in\mathcal{N}}, \mathcal{F}, \left(c_k\right)_{k\in\mathcal{N}}\right)$, where:*

- *$\left(\mathcal{N}, \left(h_k\right)_{k\in\mathcal{N}}, \mathcal{F}\right)$ is a pricing structure,*
- *and $\left(c_k\right)_{k\in\mathcal{N}} \in \mathbb{R}_{++}^{\mathcal{N}}$ is a profile of marginal costs.*

Definition 11. *The normal-form game associated with pricing game $\left(\mathcal{N}, \left(h_k\right)_{k\in\mathcal{N}}, \mathcal{F}, \left(c_k\right)_{k\in\mathcal{N}}\right)$ is triple $\left(\mathcal{F}, \left(\mathbb{R}_{++}^{f}\right)_{f\in\mathcal{F}}, \left(\pi^f\right)_{f\in\mathcal{F}}\right)$, where for every $f \in \mathcal{F}$,*

$$\pi^f(p^f, p^{-f}) = \sum_{k\in f}(p_k^f - c_k)\frac{-h_k'(p_k^f)}{\sum_{j\in f} h_j(p_j^f) + \sum_{j\in\mathcal{N}\setminus f} h_j(p_j^{-f})}, \ \forall p^f \in \mathbb{R}_{++}^{f}, \ \forall p^{-f} \in \mathbb{R}_{++}^{\mathcal{N}\setminus f}.$$

A pricing equilibrium of pricing game $\left(\mathcal{N}, \left(h_k\right)_{k\in\mathcal{N}}, \mathcal{F}, \left(c_k\right)_{k\in\mathcal{N}}\right)$ is a Nash equilibrium of game $\left(\mathcal{F}, \left(\mathbb{R}_{++}^{f}\right)_{f\in\mathcal{F}}, \left(\pi^f\right)_{f\in\mathcal{F}}\right)$.

3.2 Main Idea for the Existence Proof

Let $\left(\mathcal{N}, (h_k)_{k\in\mathcal{N}}, \mathcal{F}, (c_k)_{k\in\mathcal{N}}\right)$ be a pricing game. There are three main difficulties associated with the equilibrium existence problem. First, nothing guarantees that payoff functions are quasi-concave. In fact, it is well known that, with standard multinomial logit demands, profit functions can fail to be quasi-concave (Spady, 1984; Hanson and Martin, 1996). Second, firms' action sets are unbounded, and therefore not compact. Third, payoff functions are not necessarily supermodular. With single-product firms, it is possible to get around this issue by showing that profit functions are log-supermodular (see Milgrom and Shannon, 1994; Vives, 2000).[6] Unfortunately, we are not aware of a similar trick for multi-product firms.

The first two difficulties imply that standard existence theorems for compact games (such as Nash or Glicksberg's theorem) based on Kakutani's fixed-point theorem cannot be applied. The last two difficulties imply that existence theorems based on supermodularity theory and Tarski's fixed-point theorem (see Milgrom and Shannon, 1994; Topkis, 1998) have no bite.

The idea behind our existence proof is to exploit the fact that the pricing game is aggregative (Selten, 1970). In the following semi-formal discussion, we assume that first-order conditions are sufficient for optimality. The first-order conditions for each firm's profit maximization problem hold at price vector $p \in \mathbb{R}^{\mathcal{N}}_{++}$ if and only if for every $f \in \mathcal{F}$ and $k \in f$,

$$\frac{\partial \pi^f}{\partial p_k} = \frac{-h_k'(p_k)}{H}\left(1 - \frac{p_k - c_k}{p_k} p_k \frac{-h_k''(p_k)}{-h_k'(p_k)} + \sum_{j\in f}(p_j - c_j)\frac{-h_j'(p_j)}{H}\right) = 0,$$

where $H = \sum_{j\in\mathcal{N}} h_j(p_j)$ is the aggregator. Put $\iota_k(p_k) = p_k \frac{-h_k''(p_k)}{-h_k'(p_k)}$ for every $k \in \mathcal{N}$ and $p_k > 0$. Then, profile of prices p is a pricing equilibrium if and only if

$$\frac{p_k - c_k}{p_k}\iota_k(p_k) = 1 + \sum_{j\in f}(p_j - c_j)\frac{-h_j'(p_j)}{H}, \ \forall f \in \mathcal{F}, \ \forall k \in f \tag{4}$$

$$\text{and } H = \sum_{j\in\mathcal{N}} h_j(p_j). \tag{5}$$

Notice that, for a given $f \in \mathcal{F}$, the right-hand side does not depend on $k \in f$. It follows

[6] If $f = \{k\}$ is a single-product firm, then, for every $j \in \mathcal{N}$ such that $j \neq k$,

$$\frac{\partial^2 \log\left(\pi^f\right)}{\partial p_k \partial p_j} = \frac{h_j'(p_j) h_k'(p_k)}{\sum_{i\in\mathcal{N}} h_i(p_i)} > 0.$$

that, in any Nash equilibrium, for any $f \in \mathcal{F}$, and for all $k, l \in f$,

$$\frac{p_k - c_k}{p_k} \iota_k(p_k) = \frac{p_l - c_l}{p_l} \iota_l(p_l).$$

Put differently, there exists a real number μ^f such that $\frac{p_k - c_k}{p_k} \iota_k(p_k) = \mu^f$ for every $k \in f$.

Assume that function $p_k \mapsto \frac{p_k - c_k}{p_k} \iota_k(p_k)$ is one-to-one for every $k \in \mathcal{N}$, and denote its inverse function by $r_k(.)$. Then, firm f's strategy can be fully described by a uni-dimensional variable, μ^f, such that

$$\mu^f = 1 + \sum_{j \in f} \left(r_j(\mu^f) - c_j \right) \frac{-h'_j \left(r_j(\mu^f) \right)}{H}.$$

Assume that the above equation has a unique solution in μ^f, denoted $m^f(H)$. m^f is called firm f's fitting-in function. Then, the equilibrium existence problem boils down to finding an H such that

$$H = \underbrace{\sum_{f \in \mathcal{F}} \sum_{j \in f} h_j \left(r_j \left(m^f(H) \right) \right)}_{\equiv \Gamma(H)}.$$

In the parlance of aggregative games, Γ is called the aggregate fitting-in function. The equilibrium existence problem reduces to finding a fixed point of the aggregate fitting-in function. As we will see later on, the aggregative games approach is also useful to establish equilibrium uniqueness: the pricing game has a unique equilibrium if the following index condition is satisfied: $\Gamma'(H) < 1$ whenever $\Gamma(H) = H$.

This informal presentation leaves a number of questions open. Are first-order conditions sufficient for optimality? Is function $p_k \mapsto \frac{p_k - c_k}{p_k} \iota_k(p_k)$ one-to-one for every k? Are fitting-in functions well-defined? Does the aggregate fitting-in function have a fixed point? We need two more assumptions to answer all these questions in the affirmative.

3.3 Assumptions

In the following, we fix a pricing structure $\left(\mathcal{N}, (h_k)_{k \in \mathcal{N}}, \mathcal{F} \right)$. We make the following assumption:

Assumption 1. *For every $k \in \mathcal{N}$, function $\iota_k : x \in \mathbb{R}_{++} \mapsto x \frac{h''_k(x)}{-h'_k(x)}$ is non-decreasing.*

This assumption holds under multinomial logit and CES demands. Under logit demands ($h_i(x_i) = \mathrm{e}^{\frac{a_i - x_i}{\lambda}}$, $\lambda > 0$), $\iota_k(x) = \lambda x$ for every k. Under CES demands ($h_i(x_i) = x_i^{1-\sigma}$, $\sigma > 1$),

$\iota_k(x) = \sigma$ for every k. To gain intuition, consider the monopolistic competition benchmark with single-product firms. Under monopolistic competition, firm k does not internalize the impact of its behavior on aggregator H. It therefore maximizes $(p_k - c_k)\frac{-h_k'(p_k)}{H}$. The absolute value of the (perceived) price elasticity of firm k's demand at price p_k is therefore equal to $\iota_k(p_k)$. Assumption 1 states that the price elasticity of the monopolistic competition demand for product k is non-decreasing. If this assumption were violated, then the monopolistic competition game would have undesirable properties: an equilibrium may fail to exist; or multiple equilibria may arise; the equilibrium level of the aggregator could be discontinuous in firms' marginal costs.[7] All these issues are ruled out under Assumption 1. We will see that Assumption 1 also makes the multiproduct-firm pricing game well behaved.

Under Assumption 1, we can prove the following lemma:

Lemma 2. *For every $k \in \mathcal{N}$, $\bar{\mu}_k \equiv \lim_\infty \iota_k > 1$.*

Proof. See Appendix B.1. □

Notice that some of the $\bar{\mu}_k$'s may be infinite. We impose the following constraint on the $\bar{\mu}_k$'s:

Assumption 2.

$$\forall f \in \mathcal{F},\ \forall i, j \in f,\ \bar{\mu}_i = \bar{\mu}_j \equiv \bar{\mu}^f.$$

Assumption 2 mainly plays a technical role. We conjecture that our existence proof would still go through without it. See the discussion after Theorem 3.

3.4 Preliminary Technical Lemmas

Fix a pricing structure $\left(\mathcal{N}, (h_k)_{k\in\mathcal{N}}, \mathcal{F}\right)$, and assume that Assumptions 1 and 2 hold.

For every $k \in \mathcal{N}$, define

$$\underline{p}_k = \inf\left\{x > 0 :\ \iota_k(x) > 1\right\}.$$

By Lemma 2, $\lim_\infty \iota_k > 1$. Therefore, $\underline{p}_k$ is a well-defined a non-negative real number. In addition, $\iota_k(x) > 1$ if and only if $x > \underline{p}_k$.

Lemma 3. *For every $k \in \mathcal{N}$, for every $x > \underline{p}_k$, $\gamma_k'(x) < 0$.*

[7] Notice that, under monopolistic competition, a firm always sets the same price, no matter what the competitive environment is. This comes from the fact that the indirect utility function is additively separable in (transforms of) prices, as in Bertoletti and Etro (2015).

Proof. See Appendix B.2. □

Lemma 4. *For every $k \in \mathcal{N}$, $\lim_\infty \gamma_k = 0$.*

Proof. See Appendix B.3. □

Lemma 5. *For every $k \in \mathcal{N}$, $\lim_{x\to\infty} x h_k'(x) = 0$.*

Proof. See Appendix B.4. □

For every $k \in \mathcal{N}$ and $x > \underline{p}_k$, define

$$\rho_k(x) \equiv \frac{h_k(x)}{\gamma_k(x)}.$$

Lemma 6. *Let $k \in \mathcal{N}$, and assume that $\bar{\mu}_k < \infty$ and $\lim_\infty h_k = 0$. Then, $\lim_\infty \rho_k = \frac{\bar{\mu}_k}{\bar{\mu}_k - 1}$.*

Proof. See Appendix B.5. □

3.5 The Common ι-Markup Property

Fix a pricing game $\left(\mathcal{N}, (h_k)_{k\in\mathcal{N}}, \mathcal{F}, (c_k)_{k\in\mathcal{N}}\right)$, and assume that Assumptions 1 and 2 hold.

Definition 12. *Let $f \in \mathcal{F}$. A profile of prices for firm f, $(p_k)_{k\in f}$, satisfies the common ι-markup property if there exists $\mu^f \geq 0$ such that*

$$\forall k \in f,\ \mu^f = \frac{p_k - c_k}{p_k}\iota_k(p_k).$$

Lemma 7. *Let $f \in \mathcal{F}$ and $k \in f$. Define*

$$\nu_k :\ (p_k, c_k) \in \left\{(p_k, c_k) \in \mathbb{R}^2_{++} :\ p_k > c_k\right\} \mapsto \frac{p_k - c_k}{p_k}\iota_k(p_k).$$

For every $c_k > 0$, function $\nu_k(., c_k)$ is a strictly increasing $\mathcal{C}^1$-diffeomorphism from (c_k, ∞) to $(0, \bar{\mu}^f)$. Denote its inverse function by $r_k(., c_k)$. Then, for all $\mu^f \in (0, \bar{\mu}^f)$,

$$\frac{\partial r_k}{\partial \mu^f} = \frac{\gamma_k(r_k(\mu^f, c_k))}{\mu^f\left(-\gamma_k'\left(r_k(\mu^f, c_k)\right)\right) - (\mu^f - 1)\left(-h_k'\left(r_k(\mu^f, c_k)\right)\right)} > 0.$$

In addition, r_k is strictly increasing in μ^f and c_k, and $r_k(\mu^f, c_k) > \underline{p}_k$ whenever $\mu^f \geq 1$.

Proof. See Appendix B.6. □

In the following, we will often omit argument c_k from functions r_k and ν_k, and write r_k' instead of $\partial r_k / \partial \mu^f$ when there is no risk of confusion.

3.6 The Firm's Profit-Maximization Problem

Fix a pricing game $\left(\mathcal{N}, (h_k)_{k\in\mathcal{N}}, \mathcal{F}, (c_k)_{k\in\mathcal{N}}\right)$, and assume that Assumptions 1 and 2 hold. Let $f \in \mathcal{F}$, and fix $H^{-f} > 0$. In this subsection, we study the following maximization problem:

$$\max_{(p_k)_{k\in f}\in\mathbb{R}^f_{++}} \Pi^f\left((p_k)_{k\in f}, H^{-f}\right), \tag{6}$$

where

$$\Pi^f\left((p_k)_{k\in f}, H^{-f}\right) \equiv \sum_{k\in f}(p_k - c_k)\frac{-h'_k(p_k)}{H^{-f} + \sum_{j\in f} h_j(p_j)}.$$

Notice that, if $p^{-f} \in \mathbb{R}^{\mathcal{N}\setminus f}_{++}$ and $H^{-f} = \sum_{k\in\mathcal{N}\setminus f} h_k(p_k^{-f})$, then solving maximization problem (6) gives us firm f's best response to p^{-f}.

Let us first rule out below-cost pricing:

Lemma 8. *Let $(p_k)_{k\in f} \in \mathbb{R}^f_{++}$, and assume that there exists $k \in f$ such that $p_k < c_k$. Then, $(p_k)_{k\in f}$ is not a solution of maximization problem* (6)*.*

Proof. See Appendix B.7. □

Therefore, maximization problem (6) is equivalent to the following maximization problem (in the sense that the sets of solutions of both problems coincide):

$$\max_{(p_k)_{k\in f}\in\prod_{k\in f}[c_k,\infty)} \Pi^f\left((p_k)_{k\in f}, H^{-f}\right), \tag{7}$$

We therefore focus on maximization problem (7) from now on. In the following two lemmas, we show that if a profile of prices satisfies the first-order conditions of problem (7), then it has the common ι-markup property, and that there exists a unique profile of prices satisfying these first-order conditions.

Lemma 9. *Let $(p_k)_{k\in f} \in \prod_{k\in f}[c_k, \infty)$. The following statements are equivalent:*

(i) The first-order conditions for maximization problem (7) *hold at price profile $(p_k)_{k\in f}$.*

(ii) Price profile $(p_k)_{k\in f}$ satisfies the common ι-markup property. The corresponding μ^f solves the following equation on interval $(1, \bar{\mu}^f)$:

$$\mu^f = 1 + \mu^f \frac{\sum_{j\in f}\gamma_j\left(r_j(\mu^f)\right)}{H_{-f} + \sum_{j\in f} h_j\left(r_j(\mu^f)\right)}. \tag{8}$$

Proof. See Appendix B.8. □

Lemma 10. *Equation* (8) *has a unique solution on interval* $(1, \bar{\mu}^f)$, *which we denote by* $\hat{\mu}^f$. *Therefore, there exists a unique profile of prices that satisfies the first-order conditions of maximization problem* (7)*. In addition,*

$$\Pi^f \left(\left(r_k(\hat{\mu}^f) \right)_{j \in f}, H^{-f} \right) = \hat{\mu}^f - 1.$$

Proof. See Appendix B.9. □

Next, we show that first-order conditions are necessary for optimality:

Lemma 11. *If* $(p_k)_{k \in f}$ *solves maximization problem* (7)*, then the first-order conditions are satisfied at price profile* $(p_k)_{k \in f}$.

Proof. See Appendix B.10. □

Lemma 12. *Maximization problem* (7) *has a solution.*

Proof. See Appendix B.11. □

Combining these lemmas, we get:

Proposition 1. *There exists a unique profile of prices* $\left(p_k(H^{-f})\right)_{k \in f} \in \mathbb{R}^f_{++}$ *which solves maximization problem* (6)*. At this price profile, the first order conditions hold, and condition (ii) in Lemma 9 is satisfied.*
Conversely, if the first order conditions of maximization problem (6) *hold at price profile* $(p_k)_{k \in f} \in \prod_{k \in f} [c_k, \infty)$*, or, equivalently, if price profile* $(p_k)_{k \in f} \in \prod_{k \in f} [c_k, \infty)$ *satisfies condition (ii) in Lemma 9, then* $(p_k)_{k \in f} = \left(p_k(H^{-f})\right)_{k \in f}$.
Moreover, $\Pi^f \left(\left(p_k(H^{-f}) \right)_{k \in f}, H^{-f} \right) = \hat{\mu}^f - 1$*, where* $\hat{\mu}^f$ *is defined in Lemma 10.*

3.7 Existence of a Pricing Equilibrium

Fix a pricing game $\left(\mathcal{N}, (h_k)_{k \in \mathcal{N}}, \mathcal{F}, (c_k)_{k \in \mathcal{N}}\right)$, and assume that Assumptions 1 and 2 hold.

Lemma 13. *If* $(p^*_k)_{k \in \mathcal{N}} \in \mathbb{R}^{\mathcal{N}}_{++}$ *is a pricing equilibrium, then* $p^*_k \geq c_k$ *for every* k *in* $\mathcal{N}$.
Let $(p^*_k)_{k \in \mathcal{N}} \in \prod_{k \in \mathcal{N}} [c_k, \infty)$*. The following statements are equivalent:*

(i) $(p^*_k)_{k \in \mathcal{N}}$ *is a pricing equilibrium.*

(ii) There exist $\left(\mu^{f*}\right)_{f \in \mathcal{F}} \in \prod_{f \in \mathcal{F}} (1, \bar{\mu}^f)$ *and* $H^* > 0$ *such that*

$$p^*_k = r_k\left(\mu^{f*}\right), \ \forall f \in \mathcal{F}, \ \forall k \in f,$$

$$H^{*}=\sum_{f\in\mathcal{F}}\sum_{k\in f}h_{k}\left(r_{k}\left(\mu^{f*}\right)\right),$$

$$\mu^{f*}=1+\mu^{f*}\frac{\sum_{k\in f}\gamma_{k}\left(r_{k}(\mu^{f*})\right)}{H^{*}},\ \forall f\in\mathcal{F}.$$

Proof. See Appendix B.12. □

Therefore, looking for a pricing equilibrium boils down to looking for a pair $\left(H,(\mu^f)_{f\in\mathcal{F}}\right)\in\mathbb{R}_{++}\times\prod_{f\in\mathcal{F}}(1,\bar{\mu}^f)$ such that $H=\sum_{f\in\mathcal{F}}\sum_{k\in f}h_k\left(r_k\left(\mu^f\right)\right)$, and, for all $f\in\mathcal{F}$,

$$\mu^{f}\left(1-\frac{\sum_{k\in f}\gamma_{k}\left(r_{k}(\mu^{f})\right)}{H}\right)=1. \tag{9}$$

Lemma 14. *For every $f\in\mathcal{F}$, for every $H>0$, equation (9) has a unique solution in μ^f on interval $(1,\bar{\mu}^f)$. Denote this solution by $m^f(H)$.*
Function $m^f(.)$ is $\mathcal{C}^1$. Its derivative is given by

$$m^{f\prime}(H)=-\frac{1}{H}\frac{m^{f}(H)(m^{f}(H)-1)}{1+m^{f}(H)(m^{f}(H)-1)\frac{\sum_{k\in f}r_{k}'\left(m^{f}(H)\right)\left(-\gamma_{k}'\left(r_{k}\left(m^{f}(H)\right)\right)\right)}{\sum_{k\in\mathcal{N}^{f}}\gamma_{k}\left(r_{k}\left(m^{f}(H)\right)\right)}}<0. \tag{10}$$

Moreover, $\lim_{0^+}m^f=\bar{\mu}^f$ *and* $\lim_{\infty}m^f=1$.

Proof. See Appendix B.13. □

For every $H>0$, put

$$\Omega(H)=\frac{\sum_{f\in\mathcal{F}}\sum_{k\in f}h_{k}\left(r_{k}\left(m^{f}(H)\right)\right)}{H}.$$

By Lemmas 13 and 14, the equilibrium existence problem boils down to finding an $H\in\mathbb{R}_{++}$ such that $\Omega(H)=1$.

Lemma 15. *There exists $H^*>0$ such that $\Omega\left(H^*\right)=1$.*

Proof. See Appendix B.14. □

We can conclude:

Theorem 3. *Let $\left(\mathcal{N},(h_k)_{k\in\mathcal{N}},\mathcal{F}\right)$ be a pricing structure. If Assumptions 1 and 2 hold, then, for every $(c_k)_{k\in\mathcal{N}}\in\mathbb{R}^{\mathcal{N}}_{++}$, pricing game $\left(\mathcal{N},(h_k)_{k\in\mathcal{N}},\mathcal{F},(c_k)_{k\in\mathcal{N}}\right)$ has an equilibrium.*

In any equilibrium, the common ι-markup property is satisfied for every firm. Moreover, a firm's equilibrium profit is equal to its common ι-markup minus 1.

Therefore, if we know that H^* is an equilibrium aggregator level, then we can compute the indirect subutility of the representative consumer ($\log(H^*)$), the profit of firm $f \in \mathcal{F}$ ($m^f(H^*) - 1$) and the price of product $k \in f$ ($r_k\left(m^f(H^*)\right)$). If there are multiple equilibria, then these equilibria can be ranked: consumers prefer equilibria with higher H, whereas all firms prefer those with lower H.

Our existence theorem relies on Assumption 2, but we think the result would still go through without this assumption.[8] To fix ideas, suppose $f = \{1, 2\}$, i.e., firm f has two products, and assume that $\bar{\mu}_1 < \bar{\mu}_2$, so that Assumption 2 does not hold. The problem is that firm f might end up setting a markup $\mu^f \in (\bar{\mu}_1, \bar{\mu}_2)$, so that the price of product 1 would have to go to infinity. We therefore allow firms to set infinite prices for some of their products. We extend firm f's profit function by continuity to points (p_1, ∞), (∞, p_2) and (∞, ∞) for every $p_1, p_2 > 0$: for every $p^{-f} \in \mathbb{R}_{++}^{\mathcal{N}\setminus f}$,[9]

$$
\begin{aligned}
\pi^f\left((p_1, \infty), p^{-f}\right) &= (p_1 - c_1)\frac{-h_1'(p_1)}{h_1(p_1) + \lim_\infty h_2 + \sum_{j \notin f} h_j(p_j)},\\
\pi^f\left((\infty, p_2), p^{-f}\right) &= (p_2 - c_2)\frac{-h_2'(p_2)}{\lim_\infty h_1 + h_2(p_1) + \sum_{j \notin f} h_j(p_j)},\\
\pi^f\left((\infty, \infty), p^{-f}\right) &= 0.
\end{aligned}
$$

Then, firm f's fitting-in function, m^f, varies continuously from $\bar{\mu}_2$ to 1 as H goes from 0 to $+\infty$. Therefore the aggregate fitting-in function is still continuous. When H is in the neighborhood of 0^+, m^f is strictly greater than $\bar{\mu}_1$, so the argument developed in the proof of Lemma 15 should readily extend. This guarantees the existence of a fixed point for the aggregate fitting-in function. The monotonicity argument used in the proof of Lemma 10 should then ensure that first-order conditions are sufficient for optimality.

3.8 Comparing Equilibria

Before providing conditions under which the pricing equilibrium is unique, we briefly compare equilibria. We assume throughout that Assumptions 1 and 2 hold.

[8] There are a number of technical details we still have to check, so this paragraph should not be taken as a full-fledged formal proof.

[9] By Lemma 5, the profit function is indeed continuous at points (p_1, ∞), (∞, p_2) and (∞, ∞).

The following proposition states that equilibria can be Pareto-ranked among players (firms), with this ranking being the inverse of consumers' ranking of equilibria.

Proposition 2. *Suppose that there are two pricing equilibria with aggregators* H_1^* *and* $H_2^* > H_1^*$*, respectively. Then, each firm* $f \in \mathcal{F}$ *makes a strictly larger profit in the first equilibrium (with aggregator* H_1^**), whereas consumers' indirect utility is higher in the second equilibrium (with aggregator* H_2^**).*

Proof. To see that firm f's profit is higher in the first equilibrium, recall first from Lemma 14 that the firm's $\iota-$markup $m^f(H)$ is decreasing in the aggregator H, and second that, by Theorem 3, the firm's equilibrium profit is equal to its $\iota-$markup minus 1. As $H_1^* < H_2^*$, the result follows.

To see that consumer welfare is higher in the second equilibrium, recall from Theorem 1 that indirect subutility is of the form $\log(H_i^*) + \beta$ in equilibrium $i \in \{1, 2\}$. As $H_1^* < H_2^*$, the result follows. □

In the following proposition we investigate the comparative statics with respect to marginal costs.

Proposition 3. *Consider an increase in the marginal cost of product* $k \in f$*, offered by firm* f*. In both the equilibrium with the smallest and largest value of the aggregator* H*, this induces (i) an increase in the profit of any rival firm* $g \neq f$*, (ii) an increase in the prices of all goods, and (iii) a reduction in consumer surplus (indirect utility).*

Proof. See Appendix B.15. □

4 Equilibrium Uniqueness

4.1 Preliminaries

Let $\left(\mathcal{N}, (h_k)_{k \in \mathcal{N}}, \mathcal{F}\right)$ be a pricing structure satisfying Assumptions 1 and 2. We introduce the following notations. For every $f \in \mathcal{F}$, for every $\mu^f \in (1, \bar{\mu}^f)$,

$$\omega^f = \frac{\mu^f - 1}{\mu^f},$$

$$\bar{\omega}^f = \lim_{\mu^f \to \bar{\mu}^f} \frac{\mu^f - 1}{\mu^f},$$

and for every $k \in \mathcal{N}$, for every $x > \underline{p}_k$,

$$\chi_k(x) = \frac{\iota_k(x) - 1}{\iota_k(x)},$$
$$\theta_k(x) = \frac{-h_k'(x)}{-\gamma_k'(x)}.$$

The following lemma is useful to understand our uniqueness conditions:

Lemma 16. *Let $\left(\mathcal{N}, (h_k)_{k\in\mathcal{N}}, \mathcal{F}\right)$ be a pricing structure satisfying Assumptions 1 and 2. For every $f \in \mathcal{F}$, $\omega^f \in (0, \bar{\omega}^f)$ and $k \in f$:*

- *For every x such that $\chi_k(x) > \omega^f$, $1 - \omega^f \theta_k(x) > 0$.*
- *In particular, for every $c_k > 0$, for every $x \geq r_k\left(\frac{1}{1-\omega^f}, c_k\right)$, $1 - \omega^f \theta_k(x) > 0$.*
- *In particular, for every $x > \underline{p}_k$, $\chi_k(x)\theta_k(x) \leq 1$.*

Proof. See Appendix C.1. □

We establish equilibrium uniqueness by showing that function Ω (the aggregate fitting-in function divided by the aggregator) is strictly decreasing. The following lemma provides a sufficient condition:

Lemma 17. *Let $\left(\mathcal{N}, (h_k)_{k\in\mathcal{N}}, \mathcal{F}\right)$ be a pricing structure satisfying Assumptions 1 and 2. If, for every $f \in \mathcal{F}$,*

$$\begin{aligned} &\forall \omega^f \in (0, \bar{\omega}^f),\ \forall \left(x_k\right)_{k\in f} \in \left\{\left(x_k\right)_{k\in f} \in \mathbb{R}^f_{++} :\ \forall k \in f,\ \chi_k(x_k) > \omega^f\right\}, \\ &\left(\sum_{k\in f} \frac{\omega^f \theta_k(x_k)}{1 - \omega^f \theta_k(x_k)} \gamma_k(x_k)\right)\left(\frac{1}{\sum_{k\in f} h_k(x_k)} - \frac{\omega^f}{\sum_{k\in f}\gamma_k(x_k)}\right) < 1, \end{aligned} \tag{11}$$

or, equivalently, if[10]

$$\begin{aligned} &\forall \omega^f \in (0, \bar{\omega}^f),\ \forall \left(x_k\right)_{k\in f} \in \left\{\left(x_k\right)_{k\in f} \in \mathbb{R}^f_{++} :\ \forall k \in f,\ \chi_k(x_k) > \omega^f\right\}, \\ &\sum_{i,j\in f} \gamma_i(x_i)\gamma_j(x_j)\left(\omega^f \theta_i(x_i)\frac{1 - \omega^f \rho_j(x_j)}{1 - \omega^f \theta_i(x_i)} - \rho_i(x_i)\right) < 0, \end{aligned} \tag{12}$$

then pricing game $\left(\mathcal{N}, (h_k)_{k\in\mathcal{N}}, \mathcal{F}, (c_k)_{k\in\mathcal{N}}\right)$ has a unique equilibrium for every $(c_k)_{k\in\mathcal{N}} \in \mathbb{R}^{\mathcal{N}}_{++}$.

Proof. See Appendix C.2. □

[10] Recall that $\rho_k = \frac{h_k}{\gamma_k}$.

4.2 A First Uniqueness Theorem

As can be seen by inspecting conditions (11) and (12), whether or not the equilibrium is unique depends on whether the ρ_i's are big or small compared to the θ_j's. Let $f \in \mathcal{F}$. We introduce two new assumptions:

Assumption 3. *For every $k \in f$, ρ_k is non-decreasing on $\left(\underline{p}_k, \infty\right)$, or, equivalently,*

$$\rho_k(x) \geq \theta_k(x), \ \forall x > \underline{p}_k.$$

Since

$$\rho_k' = \frac{-\gamma_k'}{\gamma_k} \left(\rho_k - \theta_k\right),$$

both wordings are indeed equivalent.

Assumption 4.

$$\max_{k \in f} \sup_{x > \underline{p}_k} \theta_k(x) \leq \min_{k \in f} \inf_{x > \underline{p}_k} \rho_k(x).$$

Notice that Assumption 4 implies Assumption 3. Both assumptions hold under multinomial logit and CES demands. Under multinomial logit demands, $\rho_k = \theta_k = 1$; under CES demands, $\rho_k = \theta_k = \frac{\sigma}{\sigma - 1}$.

Equilibrium uniqueness follows almost immediately under Assumption 4:

Theorem 4. *Let $\left(\mathcal{N}, (h_k)_{k \in \mathcal{N}}, \mathcal{F}\right)$ be a pricing structure. If Assumptions 1 and 2 hold, and if Assumption 4 holds for every $f \in \mathcal{F}$, then pricing game $\left(\mathcal{N}, (h_k)_{k \in \mathcal{N}}, \mathcal{F}, (c_k)_{k \in \mathcal{N}}\right)$ has a unique equilibrium for every $(c_k)_{k \in \mathcal{N}} \in \mathbb{R}_{++}^{\mathcal{N}}$.*

Proof. See Appendix C.3. □

Although Assumption 4 is quite convenient for uniqueness, it is also rather strong. Start with the following definition:

Definition 13. *Let $h \in \mathbb{R}_{++}^{\mathbb{R}_{++}}$.*
We say that function h is logit on interval $I \subseteq \mathbb{R}_{++}$ if there exists $(\alpha, \lambda) \in \mathbb{R}_{++}$ such that $h(x) = \alpha \, \mathrm{e}^{-\lambda x}$ for all $x \in I$.
We say that function h is (non-homothetic) CES on interval $I \subseteq \mathbb{R}_{++}$ if there exists $(\alpha, \beta, \sigma) \in \mathbb{R}_{++} \times \mathbb{R}_{+} \times (1, \infty)$ such that $h(x) = (\alpha x + \beta)^{1-\sigma}$ for all $x \in I$.

Proposition 4. *Let $\left(\mathcal{N}, (h_k)_{k \in \mathcal{N}}, \mathcal{F}\right)$ be a pricing structure satisfying Assumptions 1 and 2. Let $f \in \mathcal{F}$, and assume that Assumption 4 holds for firm f. Let $k \in f$. Assume that $\lim_\infty h_k = 0$, and that $\lim_\infty \theta_k$ exists. Then, h_k is logit or CES on $(\underline{p}_k, \infty)$.*

Proof. See Appendix C.4. □

Proposition 5. *Let* $\left(\mathcal{N}, (h_k)_{k\in\mathcal{N}}, \mathcal{F}\right)$ *be a pricing structure satisfying Assumptions 1 and 2. Let* $f \in \mathcal{F}$*, and assume that Assumption 3 holds for firm* f*. Let* $k \in f$*. Assume that* $\bar{\mu}^f = \infty$*,* $\lim_\infty h_k = 0$*,* $\rho_k \geq 1$ *and* $\lim_\infty \theta_k$ *exists. Then,* h_k *is logit on* $(\underline{c}_k, \infty)$*.*

Proof. See Appendix C.5. □

These propositions tell us two things. First, multinomial logit and CES demands are quite special. Second, Assumption 4 (and, to some extent, Assumption 3) is quite strong when $\lim_\infty h_k = 0$. To improve on this, we will prove a uniqueness theorem which does not rely on Assumption 4.

4.3 Other Uniqueness Results

Fix a pricing structure $\left(\mathcal{N}, (h_k)_{k\in\mathcal{N}}, \mathcal{F}\right)$ satisfying Assumptions 1 and 2. We start by proving two technical lemmas, which will be useful to find upper bounds for the left-hand side of expression 12.

Lemma 18. *Let* $f \in \mathcal{F}$*. Assume that Assumption 3 holds for firm* f*, that* $\lim_\infty h_k = 0$ *for all* $k \in f$*, and that* $\bar{\mu}^f < \infty$*. Then, for every* $\omega^f \in (0, \bar{\omega}^f)$*, for every* $k \in f$*, for every* $x > 0$ *such that* $\chi_k(x) > \omega^f$*,*

$$\frac{1-\bar{\omega}^f}{\bar{\omega}^f}\frac{1}{1-\omega^f} \leq \rho_k(x) \leq \frac{1}{\bar{\omega}^f}.$$

Proof. See Appendix C.6. □

Lemma 19. *For every* $\bar{\omega} \in (0,1]$*, for every* $\omega \in (0, \bar{\omega})$*, define*

$$\phi_{\omega,\bar{\omega}} : (y,z) \in \left[\frac{1-\bar{\omega}}{\bar{\omega}}\frac{1}{1-\omega}, \frac{1}{\bar{\omega}}\right]^2 \mapsto \omega y \frac{1-\omega z}{1-\omega y} + \omega z \frac{1-\omega y}{1-\omega z} - y - z.$$

There exists a threshold $\omega^* \in (0,1)$ *(*$\omega^* \simeq 0.64$*) such that if* $\bar{\omega} \leq \omega^*$*, then* $\phi_{\omega,\bar{\omega}} \leq 0$ *for all* $\omega \in (0, \bar{\omega})$*.*

Proof. See Appendix C.7. □

Theorem 5. *Let* $\left(\mathcal{N}, (h_k)_{k\in\mathcal{N}}, \mathcal{F}\right)$ *be a pricing structure satisfying Assumptions 1 and 2. Let* $f \in \mathcal{F}$*. Assume that Assumption 3 is satisfied for firm* f*, that* $\lim_\infty h_k = 0$ *for every* $k \in f$*, and that* $\bar{\omega}^f \leq \omega^*$*. Then, condition* (12) *holds for firm* f*.*

Proof. See Appendix C.8. □

Consider the following function: $h(x) = \frac{1}{\log(1+e^x)}$. This function satisfies the integrability conditions and Assumptions 1 and 3. Assumption 4 does not hold, but $\bar{\mu} = 2$ (i.e., $\bar{\omega} = 1/2 < \omega^*$) and $\lim_\infty h = 0$. Therefore, equilibrium uniqueness follows from Theorem 5, but not from Theorem 4.

In Appendix C, we prove a number of additional uniqueness results. We first show that, under certain symmetry or monotonicity conditions, Theorem 5 extends to settings where $\lim_\infty h_k > 0$ for some k's (Propositions 15 and 16; Corollaries 1 and 2 in Section C.9). Next, under Assumption 3, we prove that the equilibrium is unique if firms' products are symmetric (Proposition 17), or if firms are single-product firms (Corollary 3). When Assumption 3 does not hold, we show that the equilibrium is unique, provided that marginal costs are high enough and that $\lim_\infty h_k > 0$ for at least one k (Proposition 18). Similarly, the equilibrium is unique, provided that $\lim_\infty \sum_{k\in\mathcal{N}} h_k$ is high enough (Proposition 19); put differently, the equilibrium is unique if the outside option is attractive enough.

4.4 A Cookbook for Constructing Pricing Structures that Yield a Unique Pricing Equilibrium

Proposition 6. *Let $\left(\mathcal{N}, (h_k)_{k\in\mathcal{N}}, \mathcal{F}\right)$ be a pricing structure satisfying Assumptions 1, 2, and 4 for every $f \in \mathcal{F}$. Assume that $\rho_k \geq 1$ for every $k \in \mathcal{N}$.*
Let $\left(\widetilde{\mathcal{N}}, \left(\widetilde{h}_k\right)_{k\in\widetilde{\mathcal{N}}}, \widetilde{\mathcal{F}}\right)$ be such that $\widetilde{\mathcal{N}}$ is a finite set containing at least two elements, $\widetilde{\mathcal{F}}$ is a partition of $\widetilde{\mathcal{N}}$ containing at least two elements, and

$$\forall g \in \widetilde{\mathcal{F}},\ \exists f \in \mathcal{F},\ \forall j \in g,\ \exists k \in f,\ \exists (\alpha_j, \beta_j, \delta_j, \epsilon_j) \in \mathbb{R}^2_{++} \times \mathbb{R}^2_{+},$$
$$\forall x > 0,\ \widetilde{h}_j(x) = \alpha_j h_k(\beta_j x + \delta_j) + \epsilon_j.$$

Then, $\left(\widetilde{\mathcal{N}}, \left(\widetilde{h}_k\right)_{k\in\widetilde{\mathcal{N}}}, \widetilde{\mathcal{F}}\right)$ is a pricing structure satisfying Assumptions 1, 2, and 4 for every $f \in \widetilde{\mathcal{F}}$. In addition, $\widetilde{\rho}_k \geq 1$ for every $k \in \widetilde{\mathcal{N}}$.

Proof. See Appendix C.11. □

If we know that function h satisfies $h > 0$, $h' < 0$, $h'' > 0$, $\iota' \geq 0$, $\rho \geq 1$ and $\sup\theta \leq \inf\rho$, then we can construct a firm's set of products by defining

$$h_k(x) = \alpha_k h(\beta_k x + \delta_k) + \epsilon_k,$$

where $(\alpha_k, \beta_k, \delta_k, \epsilon_k)_{k\in\{} \in \left(\mathbb{R}^2_{++} \times \mathbb{R}^2_{+}\right)^f$. By Proposition 6, $(h_k)_{k\in\{}$ satisfies all the assumptions we need for uniqueness. For instance, we can construct a multiproduct-firm pricing game, where firm f has CES products (firm f's base h is $h(x) = x^{1-\sigma^f}$) with heterogeneous qualities, firm f' also has CES products, potentially with $\sigma^{f'} \neq \sigma^f$, and heterogeneous qualities, firm f'' has logit products (firm f'''s base h is $h(x) = \mathrm{e}^{-x}$) with heterogeneous qualities and price sensitivity parameters, etc. Other candidates for the base h include $h(x) = \exp\left(\mathrm{e}^{-x}\right)$, $h(x) = 1 + \frac{1}{1+\mathrm{e}^{1+x}}$ and $h(x) = 1 + \frac{1}{\cosh(2+x)}$, ...

5 Multinomial Logit and CES Demands

5.1 The CES case

In this section, we study a multiproduct-firm pricing game with CES demands and heterogeneous qualities and productivities. Let $\mathcal{N}$ be a finite set containing at least two elements. For every $k \in \mathcal{N}$, for every $x > 0$, put $h_k(x) = a_k x^{1-\sigma}$, where $a_k > 0$ is the quality of product k, and $\sigma > 1$ is the elasticity of substitution. We have already shown that demand component $\left(\mathcal{N}, (h_k)_{k\in\mathcal{N}}\right)$ is smooth and integrable. Let $\left(\mathcal{N}, (h_k)_{k\in\mathcal{N}}, \mathcal{F}, (c_k)_{k\in\mathcal{N}}\right)$ be a pricing game based on CES demand component $\left(\mathcal{N}, (h_k)_{k\in\mathcal{N}}\right)$. Put $h(x) = x^{1-\sigma}$ for all $x > 0$. As discussed earlier, the corresponding ι is $\iota(x) = \sigma$, and the corresponding ρ and θ are both equal to $\frac{\sigma}{\sigma-1}$. By Theorem 4 and Proposition 6, pricing game $\left(\mathcal{N}, (h_k)_{k\in\mathcal{N}}, \mathcal{F}, (c_k)_{k\in\mathcal{N}}\right)$ has a unique equilibrium.

Firm f's fitting-in function is pinned down by equation (9), which involves functions γ_k and r_k. With CES demands, $\gamma_k(x) = \frac{\sigma-1}{\sigma} a_k x^{1-\sigma}$. Recall that r_k is the inverse function of $\nu_k : p_k \mapsto \frac{p_k - c_k}{p_k}\iota_k(p_k)$. With CES demands, $\iota_k = \sigma$, so that ν_k is just σ times the Lerner index. Therefore, for every $\mu^f \in [1, \sigma)$, $r_k(\mu^f) = \frac{c_k}{1-\frac{\mu^f}{\sigma}}$. From now on, we redefine μ^f as $\frac{\mu^f}{\sigma}$, so that μ^f is firm f's Lerner index, and takes values between $1/\sigma$ and 1. Equation (9) can then be rewritten as follows:

$$\sigma\mu^f \left(1 - \frac{\sum_{k\in f} \frac{\sigma-1}{\sigma} a_k \left(\frac{c_k}{1-\mu^f}\right)^{1-\sigma}}{H}\right) = 1. \tag{13}$$

Put $T^f = \sum_{k\in f} a_k c_k^{1-\sigma}$. Simplifying and rearranging terms in (13), we get:

$$\mu^f = \frac{1}{\sigma - (\sigma-1)\frac{T^f}{H}(1-\mu^f)^{\sigma-1}}. \tag{14}$$

It follows from Lemma 14 that equation (14) has a unique solution. This implicitly defines a function $m\left(\frac{T^f}{H}\right)$. Firm f's fitting-in function is $H \mapsto m\left(\frac{T^f}{H}\right)$. An immediate implication is that firms f and g have the same type ($T^f = T^g$) if and only if they share the same fitting-in function.

Next, we claim, that if firms f and g have the same type, then their contributions to the aggregator are the same. To see this, we introduce the following notations: for a given level of aggregator H, $s_k = \frac{a_k p_k^{1-\sigma}}{H}$ is the market share of product $k \in \mathcal{N}$, and $s^f = \sum_{k \in f} s_k$ is the market share of firm $f \in \mathcal{F}$. Then, for every $f \in \mathcal{F}$ and $k \in f$,

$$s_k = \frac{a_k \left(r_k \left(\mu^f\right)\right)^{1-\sigma}}{H} = \frac{\left(1-\mu^f\right)^{\sigma-1} a_k c_k^{1-\sigma}}{H}.$$

Therefore,

$$s^f = \sum_{k \in f} \frac{\left(1-\mu^f\right)^{\sigma-1} a_k c_k^{1-\sigma}}{H} = \frac{T^f}{H}\left(1 - m\left(\frac{T^f}{H}\right)^{\sigma-1}\right) \equiv S\left(\frac{T^f}{H}\right).$$

Firm f's market share function is $H \mapsto S\left(\frac{T^f}{H}\right)$. Therefore, firms f and g share the same market share function if and only if $T^f = T^g$. Put differently, firm f and g's contributions to the aggregator are identical if and only if they have the same type.

Recall that H is an equilibrium aggregator level if and only if $\Omega(H) = 1$, where

$$\begin{aligned}
\Omega(H) &= \frac{1}{H} \sum_{f \in \mathcal{F}} \sum_{k \in f} h_k \left(r_k \left(m^f(H)\right)\right), \\
&= \sum_{f \in \mathcal{F}} \sum_{k \in f} s_k, \\
&= \sum_{f \in \mathcal{F}} S\left(\frac{T^f}{H}\right).
\end{aligned}$$

In words, H is an equilibrium aggregator level if and only if firms' market shares add up to 1.

Last, we claim that, if firms f and g have the same type, then they earn the same profit: for every $f \in \mathcal{F}$,

$$\pi^f = \sum_{k \in f} (p_k - c_k) \frac{a_k p_k^{-\sigma}}{H},$$

$$= \sum_{k \in f} \frac{p_k - c_k}{p_k} \frac{a_k p_k^{1-\sigma}}{H},$$
$$= m\left(\frac{T^f}{H}\right) S\left(\frac{T^f}{H}\right),$$
$$\equiv \pi\left(\frac{T^f}{H}\right).$$

We summarize these findings in the following proposition:

Proposition 7. *Let f and g be two CES multiproduct firms. Put $T^f = \sum_{k \in f} a_k c_k^{1-\sigma}$ and $T^g = \sum_{k \in g} a_k c_k^{1-\sigma}$. The following assertions are equivalent:*

(i) $T^f = T^g$.

(ii) Firms f and g have the same markup fitting-in function.

(iii) Firms f and g have the same market share fitting-in function.

(iv) Firms f and g have the same profit fitting-in function.

Under CES demands, firms' types are aggregative as well. Firms f and g may differ widely in terms of product portfolios, productivity and product qualities, but if their types are the same, i.e., if $T^f = T^g$, then they share the same fitting-in functions. This implies that, no matter what the competitive environment is, these two firms will always behave in the exact same way. If we replace firm f by firm g, then the equilibrium aggregator level will not change, the behavior of firm f's rivals will not be affected, and firm g will end up charging the same markup, having the same market share, and earning the same profit as firm f. Interestingly, given a multiproduct firm f, there always exists an equivalent single-product firm. To see this, define firm $\hat{f}$ as a firm selling only one product with quality $\hat{a} = \sum_{k \in f} a_k c_k^{1-\sigma}$ and marginal cost $\hat{c} = 1$. Then, $T^f = T^{\hat{f}}$, and firms f and $\hat{f}$ are therefore equivalent.

We also obtain the following comparative statics results:

Proposition 8. *In a multiproduct-firm pricing game with CES demands,*

(i) $m', S', \pi' > 0$.

(ii) For every $f \in \mathcal{F}$, $\frac{dH^*}{dT^f}, \frac{d\mu^{f*}}{dT^f}, \frac{ds^{f*}}{dT^f}, \frac{d\pi^{f*}}{dT^f} > 0$, where superscript $*$ indicates equilibrium values, and d/dT^f is the total derivative with respect to T^f (taking into account the impact of T^f on the equilibrium aggregator level).

(iii) For every $f, g \in \mathcal{F}$, $f \neq g$, $\frac{d\mu^{g*}}{dT^f}$, $\frac{ds^{g*}}{dT^f}$, $\frac{d\pi^{g*}}{dT^f} < 0$.

Proof. See Appendix D.1. □

Point (i) says that a firm charges a high markup, has a high market share, and makes high profits if it has many products, if it is highly productive, if it sells high-quality products (high T^f), or if it operates in a less competitive environment (low H). Points (ii) and (iii) say that if firm f's type increases, then consumers benefit, firm f's markup, market share and profit increase, to the detriment of its rivals.

5.2 The Logit Case

In this section, we study a multiproduct-firm pricing game with logit demands and heterogeneous qualities and productivities. Let $\mathcal{N}$ be a finite set containing at least two elements. For every $k \in \mathcal{N}$, for every $x > 0$, put $h_k(x) = \mathrm{e}^{\frac{a_k - x}{\lambda}}$, where $a_k \in \mathbb{R}$ is the quality of product k, and $\lambda > 0$ is a substitutability parameter. We have already shown that demand component $\left(\mathcal{N}, (h_k)_{k\in\mathcal{N}}\right)$ is smooth and integrable. Let $\left(\mathcal{N}, (h_k)_{k\in\mathcal{N}}, \mathcal{F}, (c_k)_{k\in\mathcal{N}}\right)$ be a pricing game based on logit demand component $\left(\mathcal{N}, (h_k)_{k\in\mathcal{N}}\right)$. Put $h(x) = \mathrm{e}^{-x}$ for all $x > 0$. As discussed earlier, the corresponding ι is $\iota(x) = x$, and the corresponding ρ and θ are both equal to 1. By Theorem 4 and Proposition 6, pricing game $\left(\mathcal{N}, (h_k)_{k\in\mathcal{N}}, \mathcal{F}, (c_k)_{k\in\mathcal{N}}\right)$ has a unique equilibrium.

As in the previous section, we want to reexpress firm f's fitting-in function. With logit demands, $\gamma_k(x) = h_k(x)$, and $\nu_k(p_k) = \frac{p_k - c_k}{p_k}\iota_k(p_k) = \frac{p_k - c_k}{\lambda}$. Therefore, for every $\mu^f \in [1, \infty)$, $r_k(\mu^f) = \lambda\mu^f + c_k$. Equation (9) can then be rewritten as follows:

$$\mu^f \left(1 - \frac{1}{H}\sum_{k\in f} \exp\left(\frac{a_k - c_k - \lambda\mu^f}{\lambda}\right)\right) = 1. \tag{15}$$

Put $T^f = \sum_{k\in f} \exp\left(\frac{a_k - c_k}{\lambda}\right)$. Simplifying and rearranging terms in (15), we get:

$$\mu^f \left(1 - \frac{T^f}{H}\,\mathrm{e}^{-\mu^f}\right) = 1. \tag{16}$$

This uniquely pins down a function $m(.)$ such that $\mu^f = m\left(\frac{T^f}{H}\right)$. As before, define $s_k =$

$\mathrm{e}^{\frac{a_k - p_k}{\lambda}}/H$ and $s^f = \sum_{k \in f} s_k$. Then,

$$s^f = \sum_{k \in f} \frac{\mathrm{e}^{\frac{a_k - c_k - \lambda \mu^f}{\lambda}}}{H} = \frac{T^f}{H} \exp\left(-m\left(\frac{T^f}{H}\right)\right) \equiv S\left(\frac{T^f}{H}\right).$$

We can then rewrite equilibrium condition $\Omega(H) = 1$ as $\sum_{f \in \mathcal{F}} S\left(\frac{T^f}{H}\right) = 1$. In addition, it is straightforward to check that

$$\pi^f = \mu^f s^f = m\left(\frac{T^f}{H}\right) S\left(\frac{T^f}{H}\right) \equiv \pi\left(\frac{T^f}{H}\right).$$

Therefore, firms' types are still aggregative under logit demands:

Proposition 9. *Let f and g be two logit multiproduct firms. Put $T^f = \sum_{k \in f} \exp\left(\frac{a_k - c_k}{\lambda}\right)$ and $T^g = \sum_{k \in g} \exp\left(\frac{a_k - c_k}{\lambda}\right)$. The following assertions are equivalent:*

(i) $T^f = T^g$.

(ii) Firms f and g have the same markup fitting-in function.

(iii) Firms f and g have the same market share fitting-in function.

(iv) Firms f and g have the same profit fitting-in function.

We also derive the following comparative statics:

Proposition 10. *In a multiproduct-firm pricing game with logit demands,*

(i) $m', S', \pi' > 0$.

(ii) For every $f \in \mathcal{F}$, $\frac{dH^*}{dT^f}, \frac{d\mu^{f*}}{dT^f}, \frac{ds^{f*}}{dT^f}, \frac{d\pi^{f*}}{dT^f} > 0$.

(iii) For every $f, g \in \mathcal{F}$, $f \neq g$, $\frac{d\mu^{g*}}{dT^f}, \frac{ds^{g*}}{dT^f}, \frac{d\pi^{g*}}{dT^f} < 0$.

Proof. See Appendix D.2. □

To summarize, we obtain type aggregation both under CES and logit demands. With CES (resp. logit) demands, the relevant ι-markup is the Lerner index (resp. the unnormalized markup), and the relevant s^f is firm f's market share in value (resp. in volume).

5.3 An Algorithm to Solve Multiproduct Pricing Games with CES Demands

Numerically solving for the equilibrium of a multiproduct-firm pricing game in an industry with many firms and products can be a daunting task with standard methods, due to the high dimensionality of the problem. Exploiting the aggregative structure of the pricing game allows us to reduce this dimensionality tremendously: instead of solving a system of $|\mathcal{N}|$ non-linear equations in $|\mathcal{N}|$, we only need to look for and $H > 0$ such that $\Omega(H) = 1$. Of course, there usually will not be a closed-form expression for $\Omega(.)$, so we still need to approximate this function numerically. But $\Omega(H)$ is simple to compute as well, since all we need to do is solve for $|\mathcal{F}|$ separate equations, each with one unknown. Below, we describe how this general approach can be implemented to solve a multiproduct-firm pricing game with CES demands.

The algorithm use two nested loops. The inner loop computes $\Omega(H)$ for a given H. The outer loop iterates on H. We start by describing the inner loop. Fix some $H > 0$. As argued in Section 5.1, we need to compute

$$s^f = \frac{T^f}{H} \left(1 - \mu^f\right)^{\sigma - 1},$$

where μ^f is the unique solution of

$$\mu^f = \frac{1}{\sigma - (\sigma - 1) \left(\frac{T^f}{H}\right) (1 - \mu^f)^{(\sigma-1)}},$$

or, equivalently,

$$\underbrace{\mu_f \left(\sigma - (\sigma - 1)\frac{T^f}{H}(1 - \mu^f)^{\sigma-1}\right) - 1}_{\equiv \phi^f(\mu^f)} = 0. \tag{17}$$

To do so, we solve equation (9) numerically using the Newton-Raphson method. The derivative of ϕ^f can be computed analytically:

$$\phi^{f\prime}(\mu^f) = \sigma - (\sigma - 1)\frac{T^f}{H}(1 - \mu^f)^{\sigma-1} + \mu^f(\sigma - 1)^2 \frac{T^f}{H}(1 - \mu^f)^{\sigma-2},$$

so we do not need to take finite differences to compute the Newton step. The usual problem with the Newton-Raphson method is that it may fail to converge if starting values are not good enough. This is a potentially major issue, because the value of $\Omega(H)$ used by the outer loop would then be inaccurate. Fortunately, the following starting value guarantees convergence:

$$\mu_0^f = \max\left(\frac{1}{\sigma},\ 1-\left(\frac{H}{T^f}\right)^{\frac{1}{\sigma-1}}\right).$$

In fact, the Newton method converges extremely fast (usually fewer than 5 steps). Notice, in addition, that this method can be easily vectorized by stacking up the μ^f's in a vector.

The outer loop iterates on H to solve equation $\Omega(\Gamma) - 1 = 0$. This can be done by using standard derivative-based methods (we currently use Matlab's implementation of the trust-region dogleg algorithm). The Jacobian can be computed analytically:

$$\begin{aligned}\Omega'(H) &= -\sum_{g\in\mathcal{F}} \frac{T^g}{H^2} S'\left(\frac{T^g}{H}\right), \\ &= \frac{-1}{H}\sum_{g\in\mathcal{F}} \frac{\frac{T^g}{H}\left(1-\mu^g\right)^{\sigma-1}}{1+(\sigma-1)^2\frac{T^g}{H}\left(1-\mu^g\right)^{\sigma-2}\left(\mu^g\right)^2}.\end{aligned}$$

We use the value of H that would prevail under monopolistic competition as starting value ($H_0 = \sum_{f\in\mathcal{F}} T^f \left(1-\frac{1}{\sigma}\right)^{\sigma-1}$), and we always get convergence (usually in about 20 steps).[11]

6 Application to Merger Policy

We first provide a simple static merger analysis under the assumption that the merged firm continues to produce the same set of products as the merger partners did before the merger. We allow for merger-specific synergies in marginal cost, under the restriction that the merger-induced fractional change in marginal cost is the same for all products offered by the merged firm.

More formally, we consider a merger between firms f and g. Let H^* (resp., $\hat{H}^*$) denote the equilibrium value of the aggregator before (resp., after) the merger. As indirect utility is increasing in the value of that aggregator, we say that the merger is *CS-increasing* (resp., *CS-decreasing*) if $\hat{H}^* > H^*$ (resp., $\hat{H}^* < H^*$); it is *CS-neutral* if $\hat{H}^* = H^*$.

For simplicity, we assume for now that the merged firm M continues to offer each product $k \in f \cup g$ (i.e., $M = f \cup g$), and that the post-merger marginal cost of product $k \in f \cup g$ is $\hat{c}_k = (1-\eta)c_k$, where $\eta \leq 1$ is a (possibly negative) synergy parameter.

[11] In Breinlich, Nocke and Schutz (2015), we use this algorithm to calibrate an international trade model with two countries, 160 manufacturing industries, CES demands and oligopolistic competition.

Proposition 11. *In the equilibrium with the smallest (resp., largest) value of the aggregator H, there exists a cutoff $\hat{\eta} > 0$ (which may differ across equilibria) such that the merger between firms f and g is CS-increasing if and only if $\eta > \hat{\eta}$. In the absence of synergies ($\eta = 0$), the merger raises the prices of all goods as well as the joint profit of the merger partners.*

Proof. See Appendix E.1 □

That is, for a merger to be CS-increasing, the merger has to involve synergies.

6.1 Dynamic Merger Review with CES/Logit Demand

In the remainder of this section, we assume that demand is either of the CES or multinomial logit forms. As shown in Section 5, in this case an additional aggregation property obtains: a firm's product portolio (with heterogeneous qualities and marginal costs) can be fully summarized by its one-dimensional type T^f, where $T^f = \sum_{k \in f} a_k c_k^{1-\sigma}$ in the case of CES demand and $T^f = \sum_{k \in f} \exp\left(\frac{a_k - c_k}{\lambda}\right)$. For the following merger analysis, this allows us to expense with any restriction on merger-specific synergies.

Consider a merger M between firms f and g. As before, we denote by H^* and $\hat{H}^*$ the pre- and post-merger values of the aggregator, respectively. Assume that the merger partners' pre-merger types are T^f and T^g, respectively. Let T^M denote the merged firm's post-merger type. We do not impose any restriction on the merged firm's product portfolio: the set of products may or may not coincide with those previously offered by the merger partners, and for those products that the merged firm continues to produce, the qualities and marginal costs may differ arbitrarily from those pre-merger.

We have:

Lemma 20. *There exists a cutoff $\hat{T}^M > 0$ such that merger M is CS-neutral if $T^M = \hat{T}^M$, CS-increasing if $T^M > \hat{T}^M$, and CS-decreasing if $T^M < \hat{T}^M$. Moreover, if merger M is CS-nondecreasing (i.e., either CS-neutral or CS-increasing), then it is (strictly) profitable.*

Proof. See Appendix E.2. □

We now turn to studying the interaction between mergers. Consider two mergers, M_1 and M_2, and assume that these mergers are disjoint, i.e., no firm takes part in more than one merger. In the context of a homogeneous goods Cournot model, Nocke and Whinston (2010) have established that that there is a sign-preserving complementarity in the consumer surplus effect of (disjoint) mergers that share the same sign in terms of their consumer surplus effect.

The following proposition shows that this result carries over to mergers between arbitrary multiproduct firms, provided demand takes the CES or multinomial logit forms.

Proposition 12. *There is a sign-preserving complementarity in the consumer surplus effect of disjoint mergers that share the same sign in terms of their consumer surplus effect.. If merger M_i is CS-nondecreasing (and hence profitable) in isolation, it remains CS-nondecreasing (and hence profitable) if another merger M_j, $j \neq$, that is CS-nondecreasing in isolation takes place. If merger M_i is CS-decreasing in isolation, it remains CS-decreasing if another merger M_j, $j \neq$, that is CS-decreasing in isolation takes place.*

Proof. See Appendix E.3. □

Nocke and Whinston (2010)'s result on the interaction of CS-increasing and CS-decreasing mergers also carries over to our setting with multiproduct firms:

Proposition 13. *Suppose that mergers M_1 and M_2 are CS-nondecreasing and CS-decreasing, respectively, in isolation. Then, merger M_1 is CS-increasing (and hence profitable), conditional on merger M_2 taking place. Moreover, the joint profit of the firms involved in M_1 is strictly larger if both mergers take place than if neither does.*

Proof. The proof is identical to that of Proposition 2 in Nocke and Whinston (2010). It involves inverting the order of the two mergers: at the first step, merger M_2 and, at the second step, merger M_1. As consumer surplus must, by assumption, be (weakly) higher after both mergers have taken place than before, and because consumer surplus (strictly) falls at step 1 (again, by assumption), consumer surplus must (strictly) increase at step 2. That is, M_1 is CS-increasing, conditional on M_2 taking place. By Lemma 20, this implies that the joint profit of the firms in M_1 must go up at step 2. Finally, we assert that the joint profit of the firms in M_1 must go up at step 1 as well. This follows from an argument identical to that used in the proof of Proposition 2, as the CS-decreasing merger at step 1 induces a reduction in the equilibrium value of the aggregator H^*, which benefits all outsiders to that merger, including the firms involved in M_1. □

We now embed our pricing game in a dynamic model with endogenous mergers and merger policy, as in Nocke and Whinston (2010). There are T periods, and a set $\{M_1, M_2, ..., M_K\}$ of disjoint potential mergers. Merger M_k becomes feasible at the beginning of period t with probability $p_{kt} \in [0, 1]$, where $\sum_t p_{kt} \leq 1$. Conditional on becoming feasible, the post-merger type of the merged firm M_k is drawn from some distribution C_{kt}.[12] The feasibility of a

[12] The main result would not change if we assumed instead an arbitrary stochastic process governing (i) when mergers become feasible, and (ii) the merged firms' post-merger type.

particular merger (including its efficiency) is publicly observed by all firms. In each period, the firms involved in a feasible and not-yet-approved merger decide whether or not to propose their merger to the antitrust authority. We assume that bargaining is efficient so that the merger partners propose the merger if and only if it is in their joint interest to do so. Given a set of proposed mergers, the antitrust authority then decides which mergers to approve (if any). An approved merger is consummated immediately. Finally, at the end of each period, the firms play the pricing game, given current market structure. All firms as well as the antitrust authority discount payoffs with factor $\delta \leq 1$.

Following Nocke and Whinston (2010), we define a *myopically CS-maximizing merger policy* as an approval policy, where in each period, given the set of proposed mergers and current market structure, the antitrust authority approves a set of mergers that maximizes consumer surplus in the current period. The *most lenient myopically CS-maximizing merger policy* is a *myopically CS-maximizing merger policy* that approves the largest such set (i.e., including CS-neutral mergers). (As shown in Nocke and Whinston (2010) such a policy is well-defined.)

The following proposition shows that Nocke and Whinston (2010)'s result on the dynamic optimality of a myopic merger approval policy carries over to our multiproduct firm setting:

Proposition 14. *Suppose the antitrust authority adopts the most lenient myopically CS-maximizing merger policy. Then, all feasible mergers being proposed in each period after any history is a subgame-perfect Nash equilibrium for the firms. The equilibrium outcome maximizes discounted consumer surplus (indirect utility) for any realized sequence of feasible mergers. Moreover, for each such sequence, every subgame-perfect Nash equilibrium results in the same optimal sequence of period-by-period consumer surpluses.*

Proof. The result follows from Lemma 20 and Propositions 12 and 13, which are the anologs of Corollary 1 and Proposition 1 and 2 in Nocke and Whinston (2010). See Nocke and Whinston (2010) for details. □

7 Concluding Remarks

TBW

A Proofs for Section 2

A.1 Proof of Theorem 1

We first state and prove two preliminary technical lemmas, which will be useful to prove Theorem 1:

Lemma 21. *For every $n \geq 1$, for every $(\alpha_i)_{1\leq i\leq n} \in \mathbb{R}^n$, define*

$$\mathcal{M}\left((\alpha_i)_{1\leq i\leq n}\right) = \begin{pmatrix} 1-\alpha_1 & 1 & \cdots & 1 \\ 1 & 1-\alpha_2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1-\alpha_n \end{pmatrix}$$

Then,[13]

$$\det\left(\mathcal{M}\left((\alpha_i)_{1\leq i\leq n}\right)\right) = (-1)^n \left(\left(\prod_{k=1}^{n} \alpha_k\right) - \sum_{j=1}^{n} \left(\prod_{\substack{1\leq k\leq n \\ k\neq j}} \alpha_k\right)\right)$$

Moreover, matrix $\mathcal{M}\left((\alpha_i)_{1\leq i\leq n}\right)$ is negative semi-definite if and only if $\alpha_i \geq 1$ for all $1 \leq i \leq n$ and

$$\sum_{i=1}^{n} \frac{1}{\alpha_i} \leq 1.$$

Proof. We prove the first part of the lemma by induction on $n \geq 1$. Start with $n = 1$. Then,

$$\det\left(\mathcal{M}\left((\alpha_i)_{1\leq i\leq n}\right)\right) = 1-\alpha_1 = (-1)^1(\alpha_1 - 1),$$

so the property is true for $n = 1$.

Next, let $n \geq 2$, and assume the property holds for all $1 \leq m < n$. By n-linearity of the determinant,

$$\det\left(\mathcal{M}\left((\alpha_i)_{1\leq i\leq n}\right)\right) = (-\alpha_1)\begin{vmatrix} 1 & 1 & \cdots & 1 \\ 0 & 1-\alpha_2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & \cdots & 1-\alpha_n \end{vmatrix} + \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1-\alpha_2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1-\alpha_n \end{vmatrix}.$$

Applying Laplace's formula to the first column, we can see that the first determinant is,

[13] We adopt the convention that the product of an empty collection of real numbers is equal to 1.

in fact, equal to $\det\left(\mathcal{M}\left((\alpha_i)_{2\le i\le n}\right)\right)$. The second determinant can be simplified by using n-linearity one more time:

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1-\alpha_2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1-\alpha_n \end{vmatrix} = -\alpha_2 \begin{vmatrix} 1 & 0 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 1-\alpha_n \end{vmatrix} + \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1-\alpha_n \end{vmatrix},$$

$$= -\alpha_2 \det\left(\mathcal{M}\left(0,(\alpha_i)_{3\le i\le n}\right)\right) + 0,$$

where the second line follows again from Laplace's formula and from the fact that the first two rows of the second matrix in the first-line's right-hand side are colinear. Therefore,

$$\begin{aligned}
\det\mathcal{M}\left((\alpha_i)_{1\le i\le n}\right) =& -\alpha_1 \det\left(\mathcal{M}\left((\alpha_i)_{2\le i\le n}\right)\right) - \alpha_2 \det\left(\mathcal{M}\left(0,(\alpha_i)_{3\le i\le n}\right)\right), \\
=& -\alpha_1(-1)^{n-1}\left(\left(\prod_{k=2}^{n}\alpha_k\right) - \sum_{j=2}^{n}\left(\prod_{\substack{2\le k\le n\\ k\ne j}}\alpha_k\right)\right) \\
& -\alpha_2(-1)^{n-1}\left(0-\prod_{k=3}^{n}\alpha_k\right), \\
=& (-1)^n\left(\left(\prod_{k=1}^{n}\alpha_k\right) - \sum_{j=2}^{n}\left(\prod_{\substack{1\le k\le n\\ k\ne j}}\alpha_k\right) - \prod_{k=2}^{n}\alpha_k\right), \\
=& (-1)^n\left(\left(\prod_{k=1}^{n}\alpha_k\right) - \sum_{j=1}^{n}\left(\prod_{\substack{1\le k\le n\\ k\ne j}}\alpha_k\right)\right).
\end{aligned}$$

We now turn our attention to the second part of the lemma. Assume first that matrix $\mathcal{M}\left((\alpha_i)_{1\le i\le n}\right)$ is negative semi-definite. Then, all its diagonal terms have to be non-positive, i.e., $\alpha_i \ge 1$ for all i. Besides, the determinant of this matrix should be non-negative (resp. non-positive) if n is even (resp. odd). Put differently, the sign of the determinant should be $(-1)^n$ or 0. Since the α's are all different from zero, this determinant can be simplified as follows:

$$\det\left(\mathcal{M}\left((\alpha_i)_{1\le i\le n}\right)\right) = (-1)^n\left(\prod_{k=1}^{n}\alpha_k\right)\left(1-\sum_{k=1}^{n}\frac{1}{\alpha_k}\right).$$

This expression has sign $(-1)^n$ or 0 if and only if $\sum_{k=1}^{n} \frac{1}{\alpha_k} \leq 1$.

Conversely, assume that the α's are all ≥ 1, and that $\sum_{k=1}^{n} \frac{1}{\alpha_k} \leq 1$. The characteristic polynomial of matrix $\mathcal{M}\left((\alpha_i)_{1\leq i\leq n}\right)$ is defined as

$$P(X) = \begin{vmatrix} 1-\alpha_1-X & 1 & \cdots & 1 \\ 1 & 1-\alpha_2-X & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1-\alpha_n-X \end{vmatrix}.$$

This determinant can be calculated using the first part of the lemma. For every $X > 0$,

$$\begin{aligned} (-1)^n P(X) &= \underbrace{\left(\prod_{k=1}^{n}(\alpha_k + X)\right)}_{>0} \left(1 - \sum_{k=1}^{n} \frac{1}{\alpha_k + X}\right), \\ &> \underbrace{\left(\prod_{k=1}^{n}(\alpha_k + X)\right)}_{>0} \underbrace{\left(1 - \sum_{k=1}^{n} \frac{1}{\alpha_k}\right)}_{\geq 0}, \\ &> 0. \end{aligned}$$

Therefore, $P(X)$ has no strictly positive root, matrix $\mathcal{M}\left((\alpha_i)_{1\leq i\leq n}\right)$ has no strictly positive eigenvalue, and this matrix is therefore negative semi-definite. □

Lemma 22. *Let M be a symmetric n-by-n matrix, $\lambda \neq 0$, and $1 \leq k \leq n$. Let A^k be the matrix obtained by dividing the k-th line and the k-th column of M by λ. Then, M is negative semi-definite if and only if A^k is negative semi-definite.*

Proof. Suppose M is negative semi-definite, and let $X \in \mathbb{R}^n$. Write A^k as $(a_{ij})_{1\leq i,j\leq n}$ and M as $(m_{ij})_{1\leq i,j\leq n}$. Finally, define Y as the n-dimensional vector obtained by dividing X's k-th component by λ. Then,

$$\begin{aligned} X'A^kX &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_ix_j, \\ &= \left(\sum_{\substack{1\leq i\leq n \\ i\neq k}} \sum_{\substack{1\leq j\leq n \\ j\neq k}} a_{ij}x_ix_j\right) + 2x_k \sum_{\substack{1\leq i\leq n \\ i\neq k}} a_{ik}x_i + x_k^2 a_{kk}, \end{aligned}$$

$$= \left(\sum_{\substack{1 \le i \le n \\ i \neq k}} \sum_{\substack{1 \le j \le n \\ j \neq k}} m_{ij} x_i x_j \right) + 2\frac{x_k}{\lambda} \sum_{\substack{1 \le i \le n \\ i \neq k}} m_{ik} x_i + \left(\frac{x_k}{\lambda}\right)^2 m_{kk},$$

$$= \left(\sum_{\substack{1 \le i \le n \\ i \neq k}} \sum_{\substack{1 \le j \le n \\ j \neq k}} m_{ij} y_i y_j \right) + 2y_k \sum_{\substack{1 \le i \le n \\ i \neq k}} m_{ik} y_i + y^2 m_{kk},$$

$$= Y'MY,$$

$$\le 0, \text{ since } M \text{ is negative semi-definite.}$$

Therefore, A^k is negative semi-definite.

The other direction is now immediate, since M can be obtained by dividing the k-th line and the k-th column of matrix A^k by $1/\lambda$. □

We now have all we need to prove Theorem 1:

Proof. To simplify notation, assume without loss of generality that $\mathcal{N} = \{1, \ldots, n\}$, and let $D(.)$ be the demand system associated with the demand component under consideration. For every $p >> 0$, put $J(p) = \left(\frac{\partial D_i}{\partial p_j}(p)\right)_{1 \le i,j \le n}$. Theorem 1 in Nocke and Schutz (2015) states that D is quasi-linearly integrable if and only if $J(p)$ is symmetric and negative semi-definite for every $p >> 0$.

We first show that matrix $J(p)$ is symmetric for every p if and only if there exists a strictly positive scalar α such that, for every $k \in \mathcal{N}$, $g_k = -\alpha h'_k$. If $J(p)$ is symmetric for every p, then, for every $1 \le i, j \le n$ such that $i \neq j$, for every $p >> 0$,

$$-\frac{h'_j(p_j) g_i(p_i)}{\left(\sum_{k \in \mathcal{N}} h_k(p_k)\right)^2} = J_{i,j}(p) = J_{j,i}(p) = -\frac{h'_i(p_i) g_j(p_j)}{\left(\sum_{k \in \mathcal{N}} h_k(p_k)\right)^2}.$$

It follows that, for every $1 \le i \le n$, for every $x > 0$,

$$\frac{h'_i(x)}{g_i(x)} = \frac{h'_1(1)}{g_1(1)} \equiv -\beta \tag{18}$$

If $\beta = 0$, then $h'_i = 0$ for every i, which violates the assumption that h_i is strictly decreasing. Therefore, $\beta \neq 0$, and we can define $\alpha \equiv 1/\beta$. It follows that $g_i = -\alpha h'_i$. Since $g_i > 0$ and $h'_i \le 0$, we can conclude that $\alpha > 0$. Conversely, if there exists a strictly positive scalar α

such that, for every $k \in \mathcal{N}$, $g_k = -\alpha h'_k$, then, for every $1 \leq i, j \leq n$, $i \neq j$, for every $p >> 0$,

$$J_{i,j}(p) = -\frac{h'_j(p_j) g_i(p_i)}{\left(\sum_{k \in \mathcal{N}} h_k(p_k)\right)^2} = \alpha \frac{h'_j(p_j) h'_i(p_i)}{\left(\sum_{k \in \mathcal{N}} h_k(p_k)\right)^2} = J_{j,i}(p),$$

and matrix $J(p)$ is therefore symmetric for every p.

Next, suppose that there exists $\alpha > 0$ such that, for every $1 \leq k \leq n$, $g_k = -\alpha h'_k$. We want to show that $J(p)$ is negative semi-definite for every $p >> 0$ if and only if $h''_k > 0$ for every $1 \leq k \leq n$, and $\sum_{k=1}^n \gamma_k \leq \sum_{k=1}^n h_k$.

Fix $p >> 0$. To ease notation, we write $h_k = h_k(p_k)$ for every k, and define $H \equiv \sum_{k \in \mathcal{N}} h_k$. We obtain the following expression for matrix $J(p)$:

$$J(p) = \frac{\alpha}{H^2} \begin{pmatrix} (h'_1)^2 - h''_1 H & h'_1 h'_2 & \cdots & h'_1 h'_n \\ h'_2 h'_1 & (h'_2)^2 - h''_2 H & \cdots & h'_2 h'_n \\ \vdots & \vdots & \ddots & \vdots \\ h'_n h'_1 & h'_n h'_2 & \cdots & (h'_n)^2 - h''_n H \end{pmatrix}.$$

$J(p)$ is negative semi-definite if and only if

$$\begin{pmatrix} (h'_1)^2 - h''_1 H & h'_1 h'_2 & \cdots & h'_1 h'_n \\ h'_2 h'_1 & (h'_2)^2 - h''_2 H & \cdots & h'_2 h'_n \\ \vdots & \vdots & \ddots & \vdots \\ h'_n h'_1 & h'_n h'_2 & \cdots & (h'_n)^2 - h''_n H \end{pmatrix}$$

is negative semi-definite. Applying Lemma 22 n times (by dividing row k and column k by h'_k, $1 \leq k \leq n$), this is equivalent to matrix

$$\begin{pmatrix} 1 - \frac{h''_1}{(h'_1)^2} H & 1 & \cdots & 1 \\ 1 & 1 - \frac{h''_2}{(h'_2)^2} H & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 - \frac{h''_n}{(h'_n)^2} H \end{pmatrix}$$

being negative semi-definite. By Lemma 21, this holds if and only if $\frac{h''_k}{(h'_k)^2} H \geq 1$ for all $1 \leq k \leq n$, and $\frac{1}{H} \sum_{k=1}^n \frac{(h'_k)^2}{h''_k} \leq 1$. This is equivalent to $h''_k > 0$ for all k, and $\sum_{k=1}^n \gamma_k \leq \sum_{k=1}^n h_k$.

Finally, Nocke and Schutz (2015) show that, if v is such that $\nabla v = -D$, then v is an indirect subutility function for demand system D. Put $v(p) = \alpha \log\left(\sum_{j\in\mathcal{N}} h_j(p_j)\right)$. Clearly, for every $i \in \mathcal{N}$,

$$\frac{\partial v}{\partial p_i} = \frac{h_i'(p_i)}{\sum_{j\in\mathcal{N}} h_j(p_j)} = -D_i(p).$$

Therefore, v is indeed an indirect subutility function for demand system D. □

A.2 Proof of Lemma 1

Proof. For every i in $\mathcal{N}$, let $\alpha_i = \inf_{\mathbb{R}_{++}}(h_i - \gamma_i)$. Clearly, $\sum_{i\in\mathcal{N}} \alpha_i \geq 0$, otherwise it would be possible to find a price vector p such that $\sum_{i\in\mathcal{N}} (h_i(p_i) - \gamma_i(p_i)) < 0$, which would violate integrability. Let $i_0 \in \mathcal{N}$. For every i in $\mathcal{N}$, let

$$\tilde{h}_i = \begin{cases} h_i - \alpha_i & \text{if } i \neq i_0, \\ h_{i_0} - \alpha_{i_0} + \sum_{j\in\mathcal{N}} \alpha_j & \text{if } i = i_0. \end{cases}$$

Then, for every i, $\tilde{h}_i$ is $\mathcal{C}^3$, $\tilde{h}_i' = h_i' < 0$, $\tilde{h}_i'' = h_i'' > 0$, and $\tilde{\gamma}_i = \gamma_i$. In addition, for every $i \in \mathcal{N}$, for every $x > 0$,

$$\tilde{h}_i(x) - \tilde{\gamma}_i(x) \geq (h_i(x) - \gamma_i(x)) - \inf_{\mathbb{R}_{++}} (h_i - \gamma_i) \geq 0.$$

Therefore, $\tilde{h}_i \geq \tilde{\gamma}_i$ for every i. Notice, in addition, that $\tilde{h}_i \geq \tilde{\gamma}_i > 0$.

Therefore, $\left(\mathcal{N}, (\tilde{h}_i)_{i\in\mathcal{N}}\right)$ is a smooth and integrable demand component which satisfies $\tilde{h}_i \geq \tilde{\gamma}_i$ for every i. In addition, $\sum_{i\in\mathcal{N}} \tilde{h}_i = \sum_{i\in\mathcal{N}} h_i$. Therefore, for every $k \in \mathcal{N}$ and $p >> 0$,

$$\begin{aligned} D\left(\mathcal{N}, (h_j)_{j\in\mathcal{N}}\right)\left((p_j)_{j\in\mathcal{N}}\right) &= \frac{-h_k'(p_k)}{\sum_{j\in\mathcal{N}} h_j(p_j)}, \\ &= \frac{-\tilde{h}_k'(p_k)}{\sum_{j\in\mathcal{N}} \tilde{h}_j(p_j)}, \\ &= D\left(\mathcal{N}, \left(\tilde{h}_j\right)_{j\in\mathcal{N}}\right)\left((p_j)_{j\in\mathcal{N}}\right). \end{aligned}$$

Therefore, $(\mathcal{N}, (h_i)_{i\in\mathcal{N}})$ and $\left(\mathcal{N}, (\tilde{h}_i)_{i\in\mathcal{N}}\right)$ are equivalent. This concludes the proof. □

A.3 Proof of Theorem 2

Proof. Let $D \in \mathcal{D}$. There exists a smooth and integrable demand component $\left(\mathcal{N}, (h_j)_{j\in\mathcal{N}}\right)$ such that

$$D = D\left(\mathcal{N}, (h_j)_{j\in\mathcal{N}}\right).$$

By Lemma 1, there exists a smooth and integrable demand component $\left(\mathcal{N}, \left(\tilde{h}_j\right)_{j\in\mathcal{N}}\right)$ such that

$$D = D\left(\mathcal{N}, (h_j)_{j\in\mathcal{N}}\right) = D\left(\mathcal{N}, \left(\tilde{h}_j\right)_{j\in\mathcal{N}}\right),$$

and for every $k \in \mathcal{N}$, $\tilde{h}_k \geq \tilde{\gamma}_k$. For every $j \in \mathcal{N}$, put $v_j = \log\left(\tilde{h}_j\right)$. Then, since $\left(\mathcal{N}, \left(\tilde{h}_j\right)_{j\in\mathcal{N}}\right)$ is a smooth and integrable demand component, for every $j \in \mathcal{N}$, v_j is $\mathcal{C}^3$ from $\mathbb{R}_{++}$ to $\mathbb{R}$, $v_j' = \frac{\tilde{h}_j'}{\tilde{h}_j} < 0$, and

$$v_j'' = \frac{\tilde{h}_j''\tilde{h}_j - \left(\tilde{h}_j'\right)^2}{\tilde{h}_j^2}, = \frac{\tilde{h}_j''}{\tilde{h}_j^2}\left(\tilde{h}_j - \tilde{\gamma}_j\right) \geq 0.$$

Therefore, $\left(\mathcal{N}, (v_j)_{j\in\mathcal{N}}\right)$ is a discrete continuous choice model. Moreover, for every $k \in \mathcal{N}$,

$$\begin{aligned}
\Delta_k\left(\mathcal{N}, (v_j)_{j\in\mathcal{N}}\right)\left((p_j)_{j\in\mathcal{N}}\right) &= \frac{\mathrm{e}^{v_k(p_k)}}{\sum_{j\in\mathcal{N}} \mathrm{e}^{v_j(p_j)}}\left(-v_k'(p_k)\right), \\
&= \frac{-\tilde{h}_k'(p_k)}{\sum_{j\in\mathcal{N}} \tilde{h}_j(p_j)}, \\
&= D_k\left(\mathcal{N}, \left(\tilde{h}_j\right)_{j\in\mathcal{N}}\right)\left((p_j)_{j\in\mathcal{N}}\right), \\
&= D_k\left((p_j)_{j\in\mathcal{N}}\right)
\end{aligned}$$

Therefore, $D \in \mathbb{D}$.

Conversely, let $D \in \mathbb{D}$. There exists a discrete-continuous choice model $\left(\mathcal{N}, (v_j)_{j\in\mathcal{N}}\right)$ such that

$$D = \Delta\left(\mathcal{N}, (v_j)_{j\in\mathcal{N}}\right).$$

For every $j \in \mathcal{N}$, put $h_j = \mathrm{e}^{v_j}$. Then, h_j is $\mathcal{C}^3$ from $\mathbb{R}_{++}$ to $\mathbb{R}_{++}$, $h_j' = v_j' \mathrm{e}^{v_j} < 0$,

$$h_j'' = \left(\left(v_j'\right)^2 + v_j''\right)\mathrm{e}^{v_j} > 0,$$

and

$$\gamma_j = \frac{\left(h_j'\right)^2}{h_j''} = \frac{1}{1 + \frac{v_j''}{\left(v_j'\right)^2}} h_j \leq h_j.$$

Therefore, $\left(\mathcal{N}, (h_j)_{j\in\mathcal{N}}\right)$ is a smooth and integrable demand component. Moreover, for every $k \in \mathcal{N}$,

$$\begin{aligned} D_k\left(\mathcal{N}, (h_j)_{j\in\mathcal{N}}\right)\left((p_j)_{j\in\mathcal{N}}\right) &= \frac{-h_j'(p_k)}{\sum_{j\in\mathcal{N}} h_j(p_j)}, \\ &= \frac{\mathrm{e}^{v_k(p_k)}}{\sum_{j\in\mathcal{N}} \mathrm{e}^{v_j(p_j)}} \left(-v_k'(p_k)\right), \\ &= D_k\left((p_j)_{j\in\mathcal{N}}\right). \end{aligned}$$

Therefore, $D \in \mathcal{D}$. □

B Proofs for Section 3

B.1 Proof of Lemma 2

Proof. Since ι_k is monotone, $\bar{\mu}_k$ exists. Assume for a contradiction that $\bar{\mu}_k \leq 1$. In the following, drop index k to ease notation. Then, for all $x > 0$,

$$x\frac{h''(x)}{-h'(x)} \leq 1, \text{ i.e., } xh''(x) + h'(x) \leq 0.$$

It follows that

$$\frac{d}{dx}\left(xh'(x)\right) \leq 0, \quad \forall x > 0.$$

Let $x_0 > 0$. Then, for every $x \geq x_0$,

$$xh'(x) \leq x_0 h'(x_0).$$

Therefore,

$$h'(x) \leq \frac{x_0 h'(x_0)}{x}, \quad \forall x \geq x_0.$$

Integrating both sides of the inequality between x_0 and $y \geq x_0$, we get:

$$h(y) - h(x_0) \leq x_0 h'(x_0) \int_{x_0}^{y} \frac{dx}{x} \underset{y\to\infty}{\longrightarrow} -\infty, \text{ since } h' < 0.$$

Therefore, $\lim_{\infty} h = -\infty$. This contradicts the assumption that $h > 0$. $\square$

B.2 Proof of Lemma 3

Proof. Let $k \in \mathcal{N}$ and $x > \underline{p}_k$. We drop subscript k from now on to ease notation. Notice that

$$\begin{aligned}\gamma(x) &= \frac{-h'(x)}{xh''(x)} \left(x(-h'(x))\right), \\ &= \frac{-xh'(x)}{\iota(x)}.\end{aligned}$$

Therefore,

$$\begin{aligned}\gamma'(x) &= \frac{1}{\left(\iota(x)\right)^2} \left(-\left(xh''(x) + h'(x)\right) \times \iota(x) + \iota'(x) \times xh'(x)\right), \\ &= \frac{1}{\left(\iota(x)\right)^2} \left(-h'(x)\left(1 - \iota(x)\right)\iota(x) + \iota'(x)xh'(x)\right) < 0,\end{aligned}$$

as $\iota' \geq 0$ and $\iota(x) > 1$ for all $x > \underline{p}$. $\square$

B.3 Proof of Lemma 4

Proof. Fix $k \in \mathcal{N}$. We drop subscript k to ease notation. Notice first that

$$\frac{1}{\gamma} = \frac{h''}{h'^2} = \left(\frac{1}{-h'}\right)'.$$

γ is strictly positive and, by Lemma 3, strictly decreasing on $(\underline{p}, \infty)$. Therefore, $\lim_{\infty} \gamma \equiv l$ exists, and $l \in [0, \infty)$. Assume for a contradiction that $l > 0$, and let $\varepsilon > 0$. There exists $x_0 > \underline{p}$ such that

$$\forall t \geq x_0, \quad \frac{1}{\gamma(t)} \leq \frac{1}{l} + \varepsilon.$$

Integrating this inequality between x_0 and $x \geq x_0$, we get:

$$\underbrace{\left(\frac{1}{l} + \varepsilon\right)}_{\equiv M > 0}(x - x_0) \geq \int_{x_0}^{x} \frac{dt}{\gamma(t)} = \frac{1}{-h'(x)} - \underbrace{\frac{1}{-h'(x_0)}}_{\equiv A > 0}.$$

Therefore,

$$-h'(x) \geq \frac{1}{A + M(x - x_0)}, \quad \forall x \geq x_0.$$

Integrating this inequality between x_0 and $y \geq x_0$, we get:

$$\underbrace{\int_{x_0}^{y} \frac{dx}{A + M(x - x_0)}}_{\underset{y\to\infty}{\longrightarrow} \infty} \leq -h(y) + h(x_0).$$

Therefore, $\lim_\infty h = -\infty$, which contradicts the assumption that $h > 0$. □

B.4 Proof of Lemma 5

Proof. Fix $k \in \mathcal{N}$. We drop subscript k to ease notation. Notice that

$$\begin{aligned} \frac{d}{dx}\left(-xh'(x)\right) &= -h'(x) - xh''(x), \\ &= -h'(x)\left(1 - \iota(x)\right), \end{aligned}$$

which is strictly negative for all $x > \underline{p}$. Therefore, $-xh'(x)$ is strictly decreasing for x high enough, and $\lim_{x\to\infty} -xh'(x)$ is a non-negative real number. Call this number l, and assume for a contradiction that $l > 0$. Let $x_0 > \underline{p}$. Since $-xh'(x)$ is decreasing on (x_0, ∞), $-xh'(x) \geq l$ for all $x \geq x_0$. It follows that

$$h(x_0) - h(x) = \int_{x_0}^{x} -h'(t)dt \geq \underbrace{\int_{x_0}^{x} \frac{l}{t} dt}_{\underset{x\to\infty}{\longrightarrow} \infty}.$$

Therefore, $\lim_\infty h = -\infty$, which is a contradiction. □

B.5 Proof of Lemma 6

Proof. Let $k \in \mathcal{N}$ such that $\bar{\mu}_k < \infty$ and $\lim_\infty h_k = 0$. We drop subscript k from now on. For all $x > \underline{p}$,

$$\begin{aligned} \rho(x) &= \frac{h(x)h''(x)}{\left(h'(x)\right)^2}, \\ &= \frac{xh''(x)}{-h'(x)} \frac{h(x)}{-xh'(x)}, \\ &= \iota(x) \frac{h(x)}{-xh'(x)}. \end{aligned}$$

By assumption, $\lim_\infty h = 0$. By Lemma 5, $\lim_{x\to\infty} -xh'(x) = 0$. Moreover,

$$\begin{aligned}
\lim_{x\to\infty} \frac{\frac{d}{dx}h(x)}{\frac{d}{dx}\left(-xh'(x)\right)} &= \lim_{x\to\infty} \frac{h'(x)}{-h'(x) - xh''(x)}, \\
&= \lim_{x\to\infty} \frac{1}{\iota(x) - 1}, \\
&= \frac{1}{\bar{\mu} - 1}.
\end{aligned}$$

Therefore, by L'Hospital's rule,

$$\lim_{x\to\infty} \frac{h(x)}{-xh'(x)} = \frac{1}{\bar{\mu} - 1},$$

and $\lim_\infty \rho = \frac{\bar{\mu}}{\bar{\mu}-1}$. □

B.6 Proof of Lemma 7

Proof. Let $f \in \mathcal{F}$ and $k \in f$. We drop indices k and f from now on. Let $p > c$. Then, $\nu(., c)$ is $\mathcal{C}^1$, and

$$\frac{\partial \nu}{\partial p} = \underbrace{\frac{c}{p^2}\iota(p)}_{>0} + \underbrace{\frac{p-c}{p}\iota'(p)}_{\geq 0} > 0.$$

It follows from the inverse function theorem that $\nu(., c)$ is a $\mathcal{C}^1$-diffeomorphism from (c, ∞) to $\nu\left((c,\infty)\right)$. Since $\nu(., c)$ is strictly increasing,

$$\begin{aligned}
\nu\left((c,\infty), c\right) &= \left(\lim_{p\to c} \nu(p,c), \lim_{p\to\infty} \nu(p,c)\right), \\
&= (0, \bar{\mu}).
\end{aligned}$$

Notice, in addition, that

$$\nu(p,c) = (p-c)\frac{-h'(p)}{\gamma(p)}.$$

Therefore,

$$\begin{aligned}
\frac{\partial \nu}{\partial p} &= \frac{-h'(p)}{\gamma(p)} + (p-c)\frac{-h''(p)\gamma(p) + h'(p)\gamma'(p)}{\gamma(p)^2}, \\
&= \frac{-h'(p)}{\gamma(p)} + \nu(p,c)\left(-\frac{h''(p)}{-h'(p)} + \frac{-\gamma'(p)}{\gamma(p)}\right), \\
&= \frac{-h'(p)}{\gamma(p)} + \nu(p,c)\left(-\frac{-h'(p)}{\gamma(p)} + \frac{-\gamma'(p)}{\gamma(p)}\right).
\end{aligned}$$

Therefore, for all $\mu \in (0, \bar{\mu})$,

$$\frac{\partial r}{\partial \mu} = \left(\frac{\partial \nu}{\partial p} \bigg|_{(r(\mu,c),c)} \right)^{-1},$$
$$= \frac{\gamma(r(\mu, c))}{\mu \left(-\gamma' \left(r(\mu, c)\right)\right) - (\mu - 1) \left(-h' \left(r(\mu, c)\right)\right)} > 0.$$

The fact that r is increasing in c follows immediately from the fact that ν is decreasing in c and increasing in p. Finally, let us show that $r(\mu, c) > \underline{p}$ whenever $\mu \geq 1$. Let $\mu \geq 1$. If $c \geq \underline{p}$, then this is trivial. Assume instead that $c < \underline{p}$. Then,

$$\nu(\underline{p}, c) < \iota(\underline{p}) = 1 \leq \mu.$$

Since r is increasing in μ, it follows that $r(\mu, c) > \underline{p}$. □

B.7 Proof of Lemma 8

Proof. Put $g \equiv \{k \in f : \ p_k < c_k\}$, and define $(\tilde{p}_k)_{k \in f}$ as follows:

$$\tilde{p}_k = \begin{cases} c_k & \text{if } k \in g, \\ p_k & \text{otherwise.} \end{cases}$$

Since the h_ks are decreasing, $\sum_{k \in f} h_k(p_k) > \sum_{k \in f} h_k(\tilde{p}_k)$. Therefore,

$$\Pi^f \left((p_k)_{k \in f}, H^{-f} \right) = \sum_{k \in g} (p_k - c_k) \frac{-h'_k(p_k)}{H^{-f} + \sum_{j \in f} h_j(p_j)} + \sum_{k \in f \setminus g} (p_k - c_k) \frac{-h'_k(p_k)}{H^{-f} + \sum_{j \in f} h_j(p_j)},$$
$$< 0 + \sum_{k \in f \setminus g} (p_k - c_k) \frac{-h'_k(p_k)}{H^{-f} + \sum_{j \in f} h_j(p_j)},$$
$$\leq 0 + \sum_{k \in f \setminus g} (\tilde{p}_k - c_k) \frac{-h'_k(\tilde{p}_k)}{H^{-f} + \sum_{j \in f} h_j(\tilde{p}_j)},$$
$$= \Pi^f \left((\tilde{p}_k)_{k \in f}, H^{-f} \right).$$

Therefore, $(p_k)_{k \in f}$ is not a solution of maximization problem (6). □

B.8 Proof of Lemma 9

Proof. Notice first that equation (8) has no solution on interval $[0, 1]$, and that the right-hand side is not well-defined for $\mu^f \geq \bar{\mu}^f$.

The first-order conditions hold at price vector $(p_k)_{k\in f}$ if and only if, for every $k \in f$,

$$\frac{-h_k'(p_k)}{H^{-f} + \sum_{j\in f} h_j(p_j)} + (p_k - c_k)\frac{-h_k''(p_k)}{H^{-f} + \sum_{j\in f} h_j(p_j)} + \sum_{i\in f}(p_i - c_i)\frac{h_k'(p_k)h_i'(p_i)}{\left(H^{-f} + \sum_{j\in f} h_j(p_j)\right)^2} = 0.$$

Multiplying both sides of the equality by $\frac{H^{-f}+\sum_{j\in f} h_j(p_j)}{-h_k'(p_k)}$, simplifying, and rearranging terms, we see that this is equivalent to:

$$\begin{aligned}
\nu_k(p_k) &= 1 + \sum_{i\in f}(p_i - c_i)\frac{-h_i'(p_i)}{H^{-f} + \sum_{j\in f} h_j(p_j)}, \qquad (19)\\
&= 1 + \sum_{i\in f} \frac{p_i - c_i}{p_i}\frac{p_i h_i''(p_i)}{-h_i'(p_i)}\frac{\frac{\left(h_i'(p_i)\right)^2}{h_i''(p_i)}}{H^{-f} + \sum_{j\in f} h_j(p_j)},\\
&= 1 + \sum_{i\in f} \nu_i(p_i)\frac{\gamma_i(p_i)}{H^{-f} + \sum_{j\in f} h_j(p_j)}, \ \forall k \in f. \qquad (20)
\end{aligned}$$

Since the right-hand side of equation (20) does not depend on k, this is equivalent to:

$$\nu_k(p_k) = 1 + \nu_k(p_k)\frac{\sum_{j\in f}\gamma_j(p_j)}{H^{-f} + \sum_{j\in f} h_j(p_j)}, \ \forall k \in f.$$

Using the inverse functions defined in Lemma 7, this is equivalent to:

$$\begin{aligned}
p_k &= r_k(\mu^f), \ \forall k \in f,\\
\text{and } \mu^f &= 1 + \mu^f \frac{\sum_{j\in f}\gamma_j\left(r_j(\mu^f)\right)}{H^{-f} + \sum_{j\in f} h_j\left(r_j(\mu^f)\right)}.
\end{aligned}$$

□

B.9 Proof of Lemma 10

Proof. We define the following function:

$$\phi : \mu^f \in (1, \bar{\mu}^f) \mapsto (\mu^f - 1)\left(H^{-f} + \sum_{j\in f} h_j\left(r_j(\mu^f)\right)\right) - \mu^f \sum_{j\in f}\gamma_j\left(r_j(\mu^f)\right).$$

Notice that μ^f solves equation (8) if and only if $\phi(\mu^f) = 0$. ϕ is continuous, and

$$\lim_{1^+} \phi = -\sum_{j \in f} \gamma_j \left(r_j(1) \right) < 0.$$

If we show that $\lim_{\bar{\mu}^f} \phi > 0$, then we can apply the intermediate value theorem to obtain the existence of a $\mu^f \in (1, \bar{\mu}^f)$ such that $\phi(\mu^f) = 0$. Recall that $\lim_{\bar{\mu}^f} r_j = \infty$,[14] and that $\lim_\infty \gamma_j = 0$ (Lemma 4), for all $j \in f$. Notice also that, since h_j is decreasing and non-negative, $\lim_\infty h_j$ exists, and it is non-negative. We distinguish two cases. Assume first that $\bar{\mu}^f$ is finite. Then,

$$\lim_{\bar{\mu}^f} \phi = (\bar{\mu}^f - 1) \left(H^{-f} + \sum_{j \in f} \lim_\infty h_j \right) > 0.$$

Next, assume $\bar{\mu}^f = \infty$. Then,

$$\begin{aligned}
\lim_{\bar{\mu}^f} \phi &= \lim_{\mu^f \to \infty} \mu^f \left(\frac{\mu^f - 1}{\mu^f} \left(H^{-f} + \sum_{j \in f} h_j \left(r_j(\mu^f) \right) \right) - \sum_{j \in f} \gamma_j \left(r_j(\mu^f) \right) \right), \\
&= \left(\lim_{\mu^f \to \infty} \mu^f \right) \left(\lim_{\mu^f \to \infty} \left(\frac{\mu^f - 1}{\mu^f} \left(H^{-f} + \sum_{j \in f} h_j \left(r_j(\mu^f) \right) \right) - \sum_{j \in f} \gamma_j \left(r_j(\mu^f) \right) \right) \right), \\
&= \left(\lim_{\mu^f \to \infty} \mu^f \right) \left(H^{-f} + \sum_{j \in f} \lim_\infty h_j \right), \\
&= \infty.
\end{aligned}$$

Therefore, ϕ has a zero. Next, we show that ϕ is strictly increasing. By Lemma 7, ϕ is $\mathcal{C}^1$, and its derivative is given by (we omit the arguments of functions to save space):

$$\begin{aligned}
\phi'(\mu^f) &= H^{-f} + \sum_{j \in f} (h_j - \gamma_j) + (\mu^f - 1) \left(\sum_{j \in f} r'_j h'_j \right) - \mu^f \left(\sum_{j \in f} r'_j \gamma'_j \right), \\
&= H^{-f} + \sum_{j \in f} (h_j - \gamma_j) + \sum_{j \in f} \underbrace{r'_j \left(\mu^f(-\gamma'_j) - (\mu^f - 1)(-h'_j) \right)}_{= \gamma_j \text{ by Lemma 7}}, \\
&= H^{-f} + \sum_{j \in f} h_j > 0.
\end{aligned}$$

Therefore, equation (8) has a unique solution. The fact that

$$\Pi^f \left(\left(r_k(\hat{\mu}^f) \right)_{j \in f}, H^{-f} \right) = \hat{\mu}^f - 1$$

[14]This follows from the fact that r_j is strictly increasing and has range (c_k, ∞); see Lemma 7.

follows immediately from equation (19). $\square$

B.10 Proof of Lemma 11

Proof. Assume for a contradiction that $(p_k)_{k\in f}$ solves maximization problem (7), and that there exists $j \in f$ such that

$$\frac{\partial \Pi^f\left((p_k)_{k\in f}, H^{-f}\right)}{\partial p_j} \neq 0.$$

Then, $p_j = c_j$, but

$$\frac{\partial \Pi^f\left((p_k)_{k\in f}, H^{-f}\right)}{\partial p_j} = \underbrace{\frac{-h_j'(p_j)}{H^{-f}+\sum_{k\in f} h_k(p_k)}}_{>0} + \underbrace{\sum_{i\in f}(p_i - c_i)\frac{h_j'(p_j)h_i'(p_i)}{\left(H^{-f}+\sum_{k\in f} h_k(p_k)\right)^2}}_{\geq 0} > 0.$$

Therefore, $(p_k)_{k\in f}$ does not solve maximization problem (7), a contradiction. $\square$

B.11 Proof of Lemma 12

Proof. Put

$$\bar{\Pi} = \sup_{(p_k)_{k\in f}\in\prod_{k\in f}[c_k,\infty)} \Pi^f\left((p_k)_{k\in f}, H^{-f}\right).$$

There exists a sequence of price profiles $\left((p_k^n)_{k\in f}\right)_{n\in\mathbb{N}} \in \left(\prod_{k\in f}[c_k,\infty)\right)^{\mathbb{N}}$ such that

$$\Pi^f\left((p_k^n)_{k\in f}, H^{-f}\right) \underset{n\to\infty}{\longrightarrow} \bar{\Pi}.$$

To fix ideas, assume $f = \{1, \dots, N\}$. We extract subsequences as follows:

- Start with $k = 1$. Either sequence $(p_1^n)_{n\in\mathbb{N}}$ is bounded, or it is not. If it is bounded, then extract a convergent subsequence $\left(p_1^{\varphi_1(n)}\right)_{n\in\mathbb{N}}$, and denote its limit by $\hat{p}_1(\geq c_1)$. If it is unbounded, then extract a subsequence $\left(p_1^{\varphi_1(n)}\right)_{n\in\mathbb{N}}$ such that $p_1^{\varphi_1(n)} \underset{n\to\infty}{\longrightarrow} \infty$.
- Continue with $k = 2$. Either $\left(p_2^{\varphi_1(n)}\right)_{n\in\mathbb{N}}$ is bounded, or it is not. If it is bounded, then extract a convergent subsequence $\left(p_2^{\varphi_1\circ\varphi_2(n)}\right)_{n\in\mathbb{N}}$, and denote its limit by $\hat{p}_2(\geq c_2)$. If it is unbounded, then extract a subsequence $\left(p_2^{\varphi_1\circ\varphi_2(n)}\right)_{n\in\mathbb{N}}$ such that $p_2^{\varphi_1\circ\varphi_2(n)} \underset{n\to\infty}{\longrightarrow} \infty$.
- Continue this process up to $k = N$.

We are left with a subsequence of vectors $\left(p^{\varphi_1\circ\ldots\circ\varphi_N(n)}\right)_{n\in\mathbb{N}}$, which, for brevity, we relabel as $(p^n)_{n\in\mathbb{N}}$. By construction, for every k, $(p^n_k)_{n\in\mathbb{N}}$ either has a finite limit ($\hat{p}_k$), or goes to infinity as n goes to infinity.

If $(p^n_k)_{n\in\mathbb{N}}$ has a finite limit for every k, then we are done. By continuity of the profit function,

$$\bar{\Pi} = \lim_{n\to\infty} \Pi^f(p^n, H^{-f}) = \Pi^f\left((\hat{p}_k)_{1\leq j\leq N}, H^{-f}\right),$$

and $(\hat{p}_k)_{1\leq j\leq N}$ therefore maximizes $\Pi^f\left(., H^{-f}\right)$.

Next, assume for a contradiction that $(p^n_k)_{n\in\mathbb{N}}$ goes to infinity as n goes to infinity for at least one k. Assume without loss of generality that $p^n_k \underset{n\to\infty}{\longrightarrow} \infty$ for $1 \leq k \leq K$, and that $(p^n_k)_{k\in\mathbb{N}}$ has a finite limit for $K+1 \leq k \leq N$. In other words,

$$p^n \underset{n\to\infty}{\longrightarrow} (\infty, \ldots, \infty, \hat{p}_{K+1}, \ldots, \hat{p}_N) \equiv \tilde{p}.$$

Let $l_k = \lim_\infty h_k$ for every k in $\{1, \ldots, N\}$. Since the numerator and the denominator of $\Pi^f\left((p_k)_{k\in f}, H^{-f}\right)$ are additively separable in the components of price profile $(p_k)_{k\in f}$, $\lim_{(p_k)_{k\in f}\to\tilde{p}} \Pi^f\left((p_k)_{k\in f}, H^{-f}\right)$ exists and is equal to:

$$\begin{aligned}
&\frac{\left(\sum_{k=1}^K \lim_{p_k\to\infty}(p_k - c_k)\left(-h'_k(p_k)\right)\right) + \left(\sum_{k=K+1}^N \lim_{p_k\to\hat{p}_k}(p_k - c_k)\left(-h'_k(p_k)\right)\right)}{H^{-f} + \left(\sum_{k=1}^K \lim_{p_k\to\infty} h(p_k)\right) + \left(\sum_{k=K+1}^N \lim_{p_k\to\hat{p}_k} h(p_k)\right)} \\
&= \frac{\sum_{k=K+1}^N (\hat{p}_k - c_k)\left(-h'_k(\hat{p}_k)\right)}{H^{-f} + \sum_{k=1}^K l_k + \sum_{k=K+1}^N h(\hat{p}_k)},
\end{aligned}$$

since, by Lemma 5, $\lim_{x\to\infty} x h'_k(x) = \lim_{x\to\infty} h'_k(x) = 0$ for every k. It follows that

$$\begin{aligned}
\bar{\Pi} &= \lim_{n\to\infty} \Pi^f(p^n, H^{-f}), \\
&= \lim_{p\to\tilde{p}} \Pi^f(p, H^{-f}), \\
&= \frac{\sum_{k=K+1}^N (\hat{p}_k - c_k)\left(-h'_k(\hat{p}_k)\right)}{H^{-f} + \sum_{k=1}^K l_k + \sum_{k=K+1}^N h(\hat{p}_k)}.
\end{aligned}$$

If $K = N$, then $\bar{\Pi} = 0$. This is clearly a contradiction, since any price profile $(p_k)_{k\in f}$ such that $p_k > c_k$ for all k yields strictly positive profits. Next, assume that $1 < K < N$.

Claim 1: There exists $\hat{\mu}^f \in (1, \bar{\mu}^f)$ such that $\bar{\Pi} = \hat{\mu}^f - 1$.

Define a new firm, called firm $g = \{K+1, \ldots, N\}$, let

$$\Pi^g \left((p_k)_{k \in g}, H^{-g} \right) = \sum_{k \in g} (p_k - c_k) \frac{-h'_k(p_k)}{H^{-g} + \sum_{j \in g} h(p_j)},$$

and notice that

$$\Pi^g \left((\hat{p}_{K+1}, \ldots, \hat{p}_N), H^{-f} + \sum_{k=1}^{K} l_k \right) = \bar{\Pi}.$$

We claim that price vector $(\hat{p}_{K+1}, \ldots, \hat{p}_N)$ solves maximization problem

$$\max_{(p_k)_{k \in g} \in \prod_{k \in g} [c_k, \infty)} \Pi^g \left((p_k)_{k \in g}, H^{-f} + \sum_{k=1}^{K} l_k \right). \tag{21}$$

Assume this is not the case. Then, there exists $\left(p'_{K+1}, \ldots, p'_N\right)$ such that

$$\Pi^g \left(\left(p'_{K+1}, \ldots, p'_N\right), H^{-f} + \sum_{k=1}^{K} l_k \right) > \bar{\Pi}.$$

Since

$$\Pi^g \left(\left(p'_{K+1}, \ldots, p'_N\right), H^{-f} + \sum_{k=1}^{K} l_k \right) = \lim_{p \to \left(\infty, \ldots, \infty, p'_{K+1}, \ldots, p'_N\right)} \Pi^f \left(p, H^{-f}\right),$$

there exists $(p'_1, \ldots, p'_K)$ such that

$$\Pi^f \left(\left(p'_1, \ldots, p'_K, p'_{K+1}, \ldots, p'_N\right), H^{-f} \right) > \bar{\Pi},$$

which is a contradiction. Therefore, $(\hat{p}_{K+1}, \ldots, \hat{p}_N)$ solves maximization problem (21), which is the counterpart of maximization problem (7) for firm g. By Lemmas 9, 10 and 11, there exists $\hat{\mu}^f \in (1, \bar{\mu}^g) = (1, \bar{\mu}^f)$ such that $\bar{\Pi} = \hat{\mu}^f - 1$.

Claim 2: There exists $\hat{p}_K > c_K$ such that $\phi(\hat{p}_K) > \bar{\Pi}$, where

$$\phi(x) = \frac{(x - c_K)\left(-h'_K(x)\right) + \sum_{k=K+1}^{N} (\hat{p}_k - c_k)\left(-h'_k(\hat{p}_k)\right)}{H^{-f} + \left(\sum_{k=1}^{K-1} l_k\right) + h_K(x) + \sum_{k=K+1}^{N} h(\hat{p}_k)}, \forall x > c_K.$$

Notice that $\lim_\infty \phi = \bar{\Pi} = \hat{\mu}^f - 1$. Let us show that ϕ is strictly decreasing when x is high enough. The derivative of ϕ with respect to x is equal to:

$$\begin{aligned}\phi'(x) = &\Bigg\{(-h_K'(x) - (x - c_K)h_K''(x)) \left(H^{-f} + \left(\sum_{k=1}^{K-1} l_k\right) + h_K(x) + \sum_{k=K+1}^{N} h_k(\hat{p}_k)\right) \\ &- h_K'(x) \left((x - c_K)(-h_K'(x)) + \sum_{k=K+1}^{N} (\hat{p}_k - c_k)(-h_k'(\hat{p}_k))\right)\Bigg\} \\ &/ \left(H^{-f} + \left(\sum_{k=1}^{K-1} l_k\right) + h_K(x) + \sum_{k=K+1}^{N} h_k(\hat{p}_k)\right)^2. \\ = &\underbrace{\frac{-h_K'(x)}{H^{-f} + \left(\sum_{k=1}^{K-1} l_k\right) + h_K(x) + \sum_{k=K+1}^{N} h_k(\hat{p}_k)}}_{>0} (\phi(x) - (\nu_K(x) - 1)).\end{aligned}$$

Since

$$(\phi(x) - (\nu_K(x) - 1)) \underset{x\to\infty}{\longrightarrow} \begin{cases} \hat{\mu}^f - \bar{\mu}^f < 0 & \text{if } \bar{\mu}^f < \infty, \\ -\infty & \text{if } \bar{\mu}^f = \infty, \end{cases}$$

it follows that $\phi'(x) < 0$ for x high enough. Therefore, $\phi(x) > \bar{\Pi} = \lim_\infty \phi$ for x high enough, which concludes the proof of Claim 2.

We can conclude: by Claim 2,

$$\lim_{p \to (\infty,\ldots,\infty,\hat{p}_K,\hat{p}_{K+1},\ldots,\hat{p}_N)} \Pi^f\left(p, H^{-f}\right) = \phi(\hat{p}_K) > \bar{\Pi}.$$

Therefore, there exists $(\hat{p}_1, \ldots, \hat{p}_{K-1})$ such that

$$\Pi^f\left((\hat{p}_1, \ldots, \hat{p}_{K-1}, \hat{p}_K, \hat{p}_{K+1}, \ldots, \hat{p}_N), H^{-f}\right) > \bar{\Pi},$$

which is a contradiction. □

B.12 Proof of Lemma 13

Proof. The fact that firms do not price below cost in any pricing equilibrium follows immediately from Lemma 8.

$(p_k^*)_{k\in\mathcal{N}} \in \prod_{k\in\mathcal{N}}[c_k,\infty)$ is a pricing equilibrium if and only if for every $f \in \mathcal{F}$,

$$(p_k)_{k\in f} \in \arg\max_{(p_k)_{k\in f}\in\prod_{k\in f}[c_k,\infty)} \Pi^f\left((p_k)_{k\in f}, \sum_{\substack{g\in\mathcal{F}\\ g\neq f}}\sum_{k\in g} h_k(p_k^*)\right).$$

By Proposition 1, this holds if and only if for every $f \in \mathcal{F}$, there exists $\mu^{f*} \in \left(1, \bar{\mu}^f\right)$ such that

$$\begin{aligned}
&\mu^{f*} = \nu_k(p_k^*), \ \forall k \in f, \\
\text{and } &\mu^{f*} = 1 + \mu^{f*}\frac{\sum_{k\in f}\gamma_k\left(r_k(\mu^{f*})\right)}{\left(\sum_{k\in f} h_k\left(r_k\left(\mu^{f*}\right)\right)\right) + \left(\sum_{\substack{g\in\mathcal{F}\\ g\neq f}}\sum_{k\in g} h_k(p_k^*)\right)},
\end{aligned}$$

which is clearly equivalent to point (ii) in the statement of the lemma. □

B.13 Proof of Lemma 14

Proof. Let $H > 0$ and $f \in \mathcal{F}$. The left-hand side of equation (9) is continuous in μ^f, converges to

$$1 - \frac{\sum_{k\in f}\gamma_k(r_k(1))}{H} < 1$$

when μ^f goes to 1 (by Lemma 7), and converges to $\bar{\mu}^f > 1$ when μ^f goes to $\bar{\mu}^f$ (by Lemmas 4 and 7). Therefore, by the intermediate value theorem, equation (9) has a solution. In addition, by Lemmas 3 and 7, the left-hand side is strictly increasing in μ^f. Therefore, this solution is unique.

Totally differentiating equation (9), we get (we omit the arguments of functions to save space):

$$\begin{aligned}
0 &= \left(1 - \frac{\sum_{k\in f}\gamma_k}{H} + m^f\frac{\sum_{k\in f} r_k'\left(-\gamma_k'\right)}{H}\right)dm^f + m^f\frac{\sum_{k\in f}\gamma_k}{H^2}dH, \\
&= \underbrace{\left(\frac{1}{m^f} + (m^f - 1)\frac{\sum_{k\in f} r_k'\left(-\gamma_k'\right)}{\sum_{k\in f}\gamma_k}\right)}_{>0} dm^f + (m^f - 1)\frac{dH}{H},
\end{aligned}$$

as $H = \frac{m^f}{m^f - 1} \sum_{k\in f} \gamma_k$ from (9). By the implicit function theorem,

$$m^{f\prime}(H) = -\frac{1}{H} \frac{m^f(m^f-1)}{1 + m^f(m^f-1)\frac{\sum_{k\in f} r_k'(-\gamma_k')}{\sum_{k\in\mathcal{N}^f} \gamma_k}},$$

which is indeed strictly negative. It follows that m^f is strictly decreasing, and that $\lim_{0^+} m^f$ and $\lim_\infty m^f$ exist. Assume for a contradiction that $\lim_{0^+} m^f \in [1, \bar{\mu}^f)$. Then, the left-hand side of equation (9) goes to $-\infty$ as H goes to 0, which is a contradiction. Therefore, $\lim_{0^+} m^f = \bar{\mu}^f$. Next, notice that $\sum_{k\in f} \gamma_k \left(r_k(\mu^f)\right)$ is bounded when $\mu^f \geq 1$. Therefore, the left-hand side of equation (9) goes to μ^f when H goes to ∞, and $\lim_\infty m^f = 1$. □

B.14 Proof of Lemma 15

Proof. By Lemma 1, there exists a smooth and integrable demand component $\left(\mathcal{N}, \left(\tilde{h}_k\right)_{k\in\mathcal{N}}\right)$ such that $\left(\mathcal{N}, (h_k)_{k\in\mathcal{N}}\right)$ and $\left(\mathcal{N}, \left(\tilde{h}_k\right)_{k\in\mathcal{N}}\right)$ are equivalent and $\tilde{h}_k \geq \tilde{\gamma}_k$ for every $k \in \mathcal{N}$. Notice, in addition, that $\left(\mathcal{N}, \left(\tilde{h}_k\right)_{k\in\mathcal{N}}\right)$ satisfies Assumptions 1 and 2. In the following, we use $\left(\mathcal{N}, \left(\tilde{h}_k\right)_{k\in\mathcal{N}}\right)$ instead of $\left(\mathcal{N}, (h_k)_{k\in\mathcal{N}}\right)$, which is not a problem, since these demand components are equivalent. We drop the tildes to ease notation.

Ω is continuous on $\mathbb{R}_{++}$. By Lemmas 7 and 14, when H goes to ∞, the numerator of Ω goes to

$$\sum_{f\in\mathcal{F}} \sum_{k\in f} h_k\left(r_k(1)\right),$$

which is finite. Therefore, $\lim_\infty \Omega = 0$. If we show that Ω is strictly greater than 1 in the neighborhood of 0^+, then we can apply the intermediate value theorem to obtain the existence of H^*.

Assume first that there exists $j \in \mathcal{N}$ such that $\lim_\infty h_j = l > 0$. Since h_j is decreasing, $h_j(x) \geq l$ for all $x > 0$. Let $f \in \mathcal{F}$ such that $j \in f$. Then, for all $H > 0$,

$$\begin{aligned} \Omega(H) &\geq \frac{h_j\left(r_j\left(m^f(H)\right)\right)}{H}, \\ &\geq \frac{l}{H} \underset{H\to 0^+}{\longrightarrow} \infty. \end{aligned}$$

Therefore, $\lim_{0^+} \Omega = \infty > 1$.

Next, assume that $h_k(x) \underset{x\to\infty}{\longrightarrow} 0$ for all $k \in \mathcal{N}$. Put

$$\mathcal{F}' = \left\{ f \in \mathcal{F} : \ \bar{\mu}^f = \infty \right\}$$

and $\mathcal{F}'' = \mathcal{F} \backslash \mathcal{F}'$. Let $f \in \mathcal{F}''$. Then, by Lemma 6, $\lim_\infty \rho_k = \frac{\bar{\mu}^f}{\bar{\mu}^f - 1}$ for every $k \in f$. In addition, by Lemmas 7 and 14, for every $k \in f$, $r_k\left(m^f(H)\right) \underset{H\to 0^+}{\longrightarrow} \infty$. Therefore, there exists $\eta^f > 0$ such that for every $H < \eta^f$,

$$\rho_k\left(r_k\left(m^f(H)\right)\right) \geq \frac{\bar{\mu}^f}{\bar{\mu}^f - 1}\left(1 - \frac{1}{2|\mathcal{F}|}\right).$$

Let $\eta = \min_{f\in\mathcal{F}''} \eta^f$ (or any strictly positive real number if $\mathcal{F}''$ is empty). For every $H < \eta$,

$$\begin{aligned}
\Omega(H) &= \frac{\left(\sum_{f\in\mathcal{F}'}\sum_{k\in f} h_k\left(r_k\left(m^f(H)\right)\right)\right) + \left(\sum_{f\in\mathcal{F}''}\sum_{k\in f} h_k\left(r_k\left(m^f(H)\right)\right)\right)}{H}, \\
&\geq \frac{\left(\sum_{f\in\mathcal{F}'}\sum_{k\in f} \gamma_k\left(r_k\left(m^f(H)\right)\right)\right) + \left(\sum_{f\in\mathcal{F}''}\sum_{k\in f} \gamma_k\left(r_k\left(m^f(H)\right)\right)\rho_k\left(r_k\left(m^f(H)\right)\right)\right)}{H}, \\
&\geq \sum_{f\in\mathcal{F}'} \frac{\sum_{k\in f}\gamma_k}{H} + \sum_{f\in\mathcal{F}''} \frac{\bar{\mu}^f}{\bar{\mu}^f - 1}\left(1 - \frac{1}{2|\mathcal{F}|}\right)\frac{\sum_{k\in f}\gamma_k}{H}, \quad \text{since } H < \eta, \\
&= \sum_{f\in\mathcal{F}'} \frac{m^f(H) - 1}{m^f(H)} + \sum_{f\in\mathcal{F}''} \frac{m^f(H) - 1}{m^f(H)}\frac{\bar{\mu}^f}{\bar{\mu}^f - 1}\left(1 - \frac{1}{2|\mathcal{F}|}\right), \text{ using equation (9).}
\end{aligned}$$

When H goes to 0^+, the right-hand side term on the last line goes to

$$|\mathcal{F}'| + |\mathcal{F}''|\left(1 - \frac{1}{2|\mathcal{F}|}\right) \geq |\mathcal{F}| - \frac{1}{2},$$

which is strictly greater than 1. Therefore, $\Omega(H) > 1$ when H is small enough. This concludes the proof. □

B.15 Proof of Lemma 3

Consider an increase in marginal cost c_k, $k \in f$.

First, recall from Lemma 7 that $\partial r_k(\mu^f, c_k)/\partial c_k > 0$ whereas $\partial r_j(\mu^g, c_j)/\partial c_k = 0$ for all $g \in \mathcal{F}$ and $j \in g$, $j \neq k$. Next, by Lemma 14, firm f's fitting in function $m^f(H)$ is uniquely determined by

$$m^f\left(1 - \frac{\sum_{j\in f}\gamma_j\left(r_j(m^f, c_j)\right)}{H}\right) = 1.$$

Applying the implicit funtion theorem, we obtain

$$\frac{dm^f(H)}{dc_k} = \frac{m^f - \gamma_k' \frac{\partial r_k}{\partial c_k}}{H - \sum_{j\in f}\gamma_j - m^f \sum_{j\in f}\gamma_j' \frac{\partial r_j}{\partial m^f}} < 0,$$

where the inequality follows from $\partial r_k(\mu^f, c_k)/\partial c_k > 0$, $\partial r_j(m^f, c_j)/\partial m^f > 0$, and $\gamma_j'(r_j) < 0$. Hence, $\sum_{j\in f} h_j(r_j(m^f(H)))$ decreases with c_k. For any firm $g \neq f$, $dm^g(H)/dc_k = 0$, so that $\sum_{g\in\mathcal{F},g\neq f}\sum_{j\in g} h_j(r_j(m^g(H)))$ remains unchanged. This implies that, for any $H > 0$, $\Omega(H)$ is decreasing with c_k. Recall from the proof of Lemma 15 that $\Omega(H) > 1$ for H sufficiently small and $\Omega(H) < 1$ for H sufficiently large. Hence, both the smallest and the largest solution in H to $\Omega(H) = 1$ are decreasing with c_k, and so is therefore indirect utility and the profit of any firm $g \neq f$, in both the smallest and largest equilibrium. As $dm^g(H)/dH < 0$ by Lemma 14, it also follows that all prices increase in both the smallest and largest equilibrium.

C Proofs for Section 4

C.1 Proof of Lemma 16

Proof. Let $f \in \mathcal{F}$, $k \in f$, $\omega^f \in \left(0, \bar{\omega}^f\right)$, and x such that $\chi_k(x) > \omega^f$. Put $\mu^f = \frac{1}{1-\omega^f}$. Then, $\iota_k(x) > \mu^f$. Therefore, there exists $c > 0$ such that $\nu_k(x,c) = \mu^f$. We know from Lemma 7 that

$$\begin{aligned}\frac{\partial r_k}{\partial \mu^f}(\mu^f, c) &= \frac{\gamma_k(r_k(\mu^f, c_k))}{\mu^f\left(-\gamma_k'\left(r_k(\mu^f, c_k)\right)\right) - \left(\mu^f - 1\right)\left(-h_k'\left(r_k(\mu^f, c_k)\right)\right)}, \\ &= \frac{\gamma_k(x)}{-\gamma_k'(x)\mu^f}\frac{1}{1-\omega^f\theta_k(x)} > 0.\end{aligned}$$

In addition, by Lemmas 3 and 7, $\gamma_k'(x) < 0$. Therefore, $1 - \omega^f\theta_k(x) > 0$. This establishes the first bullet point in the statement of the lemma.

Next, let $c > 0$ and $x \geq r_k(\mu^f, c)$. Then, since $\nu_k(., c)$ is increasing, $\nu_k(x, c) \geq \mu^f$. Since $c > 0$, it follows that $\iota_k(x) > \mu^f$, and that $\chi_k(x) > \omega^f$. It follows from the first part of the lemma that $1 - \omega^f\theta_k(x) > 0$.

Finally, let $x > \underline{p}_k$. Put $\omega^f = \chi_k(x)$. Then, for every y such that $\chi_k(y) > \omega^f$, $1-\omega^f\theta_k(y) > 0$. By monotonicity of χ_k, this implies that, for every $y > x$, $1 - \chi_k(x)\theta_k(y) > 0$. Therefore, by continuity of θ_k, $\chi_k(x)\theta_k(x) \leq 1$. □

C.2 Proof of Lemma 17

We start by proving two intermediate lemmas:

Lemma 23. *Let* $\left(\mathcal{N},(h_k)_{k\in\mathcal{N}},\mathcal{F},(c_k)_{k\in\mathcal{N}}\right)$ *be a pricing game satisfying Assumptions 1 and 2. If, for every* $f\in\mathcal{F}$,

$$\forall\omega^f\in(0,\bar{\omega}^f),\ \left(\sum_{k\in f}\frac{\omega^f\theta_k}{1-\omega^f\theta_k}\gamma_k\right)\left(\frac{1}{\sum_{k\in f}h_k}-\frac{\omega^f}{\sum_{k\in f}\gamma_k}\right)<1, \tag{22}$$

where, for every k, *functions* θ_k, γ_k *and* h_k *are all evaluated at point* $p_k=r_k\left(\frac{1}{1-\omega^f}\right)$, *then the pricing game has a unique equilibrium.*

Proof. By Theorem 3, $\left(\mathcal{N},(h_k)_{k\in\mathcal{N}},\mathcal{F},(c_k)_{k\in\mathcal{N}}\right)$ has a pricing equilibrium. To prove that there is only one equilibrium, we will show that $\Omega(.)$ is strictly decreasing.[15] Let $H>0$, and, for every $f\in\mathcal{F}$, $\mu^f=m^f(H)$ and $\omega^f=\frac{\mu^f-1}{\mu^f}$. Then,

$$\begin{aligned}
H^2\Omega'(H) &= H\sum_{f\in\mathcal{F}}m^{f\prime(H)}\sum_{k\in f}r_k'\left(\mu^f\right)h_k'\left(r_k\left(\mu^f\right)\right)-\sum_{k\in\mathcal{N}}h_k\left(r_k\left(\mu^f\right)\right),\\
&=\sum_{f\in\mathcal{F}}\left(\frac{\mu^f(\mu^f-1)}{1+\mu^f(\mu^f-1)\frac{\sum_{k\in f}r_k'\left(-\gamma_k'\right)}{\sum_{k\in f}\gamma_k}}\left(\sum_{k\in f}r_k'\left(-h_k'\right)\right)-\sum_{k\in f}h_k\right),\ \text{by Lemma 14.}
\end{aligned}$$

Therefore, a sufficient condition for this derivative to be strictly negative is that, for all $f\in\mathcal{F}$,

$$\frac{\mu^f(\mu^f-1)\frac{\sum_{k\in f}r_k'\left(-h_k'\right)}{\sum_{k\in f}h_k}}{1+\mu^f(\mu^f-1)\frac{\sum_{k\in f}r_k'\left(-\gamma_k'\right)}{\sum_{k\in f}\gamma_k}}<1. \tag{23}$$

Let $f\in\mathcal{F}$. Then,

$$\begin{aligned}
(23) &\iff (\mu^f-1)\left(\frac{\sum_{k\in f}\mu^f r_k'\left(-h_k'\right)}{\sum_{k\in f}h_k}-\frac{\sum_{k\in f}\mu^f r_k'\left(-\gamma_k'\right)}{\sum_{k\in f}\gamma_k}\right)<1,\\
&\iff (\mu^f-1)\left(\frac{\sum_{k\in f}\mu^f\frac{\gamma_k\left(-h_k'\right)}{\mu^f\left(-\gamma_k'\right)-(\mu^f-1)\left(-h_k'\right)}}{\sum_{k\in f}h_k}-\frac{\sum_{k\in f}\mu^f\frac{\gamma_k\left(-\gamma_k'\right)}{\mu^f\left(-\gamma_k'\right)-(\mu^f-1)\left(-h_k'\right)}}{\sum_{k\in f}\gamma_k}\right)<1,
\end{aligned}$$

[15] An a priori less restrictive way of establishing uniqueness would be to show that $\Omega'(H)<1$ whenever $\Omega(H)=1$. Unfortunately, this is not particularly useful, because we simply do not know what these equilibrium H's could be. See also Appendix C.12, where we use an index approach to derive uniqueness conditions.

$$\iff (\mu^f - 1)\left(\frac{\sum_{k\in f}\frac{\theta_k}{1-\omega^f\theta_k}\gamma_k}{\sum_{k\in f} h_k} - \frac{\sum_{k\in f}\frac{1}{1-\omega^f\theta_k}\gamma_k}{\sum_{k\in f}\gamma_k}\right) < 1,$$

$$\iff (\mu^f - 1)\left(\frac{\sum_{k\in f}\frac{\theta_k}{1-\omega^f\theta_k}\gamma_k}{\sum_{k\in f} h_k} - 1 - \frac{\sum_{k\in f}\frac{\omega^f\theta_k}{1-\omega^f\theta_k}\gamma_k}{\sum_{k\in f}\gamma_k}\right) < 1,$$

$$\iff (\mu^f - 1)\left(-1 + \sum_{k\in f}\frac{\theta_k}{1-\omega^f\theta_k}\gamma_k\left(\frac{1}{\sum_{k\in f} h_k} - \frac{\omega^f}{\sum_{k\in f}\gamma_k}\right)\right) < 1,$$

$$\iff \left(\sum_{k\in f}\frac{\omega^f\theta_k}{1-\omega^f\theta_k}\gamma_k\right)\left(\frac{1}{\sum_{k\in f} h_k} - \frac{\omega^f}{\sum_{k\in f}\gamma_k}\right) < 1,$$

where, for every $k \in f$, functions θ_k, γ_k and h_k are evaluated at point $p_k = r_k(\mu^f) = r_k\left(\frac{1}{1-\omega^f}\right)$. Since condition (22) holds by assumption, Ω is strictly decreasing. Therefore, the pricing game has a unique equilibrium. □

Lemma 24. *Let* $\left(\mathcal{N}, (h_k)_{k\in\mathcal{N}}, \mathcal{F}\right)$ *be a pricing structure satisfying Assumptions 1 and 2. Let* $(\underline{c}_k)_{k\in\mathcal{N}} \in \mathbb{R}^{\mathcal{N}}_{++}$. *If, for every* $f \in \mathcal{F}$,

$$\begin{gathered}\forall \omega^f \in (0, \bar{\omega}^f),\ \forall (x_k)_{k\in f} \in \prod_{k\in f}\left[r_k\left(\frac{1}{1-\omega^f}, \underline{c}_k\right), \infty\right), \\ \left(\sum_{k\in f}\frac{\omega^f\theta_k(x_k)}{1-\omega^f\theta_k(x_k)}\gamma_k(x_k)\right)\left(\frac{1}{\sum_{k\in f} h_k(x_k)} - \frac{\omega^f}{\sum_{k\in f}\gamma_k(x_k)}\right) < 1,\end{gathered} \tag{24}$$

or, equivalently, if

$$\begin{gathered}\forall \omega^f \in (0, \bar{\omega}^f),\ \forall (x_k)_{k\in f} \in \prod_{k\in f}\left[r_k\left(\frac{1}{1-\omega^f}, \underline{c}_k\right), \infty\right), \\ \sum_{i,j\in f}\gamma_i(x_i)\gamma_j(x_j)\left(\omega^f\theta_i(x_i)\frac{1-\omega^f\rho_j(x_j)}{1-\omega^f\theta_i(x_i)} - \rho_i(x_i)\right) < 0,\end{gathered} \tag{25}$$

then pricing game $\left(\mathcal{N}, (h_k)_{k\in\mathcal{N}}, \mathcal{F}, (c_k)_{k\in\mathcal{N}}\right)$ *has a unique equilibrium for every* $(c_k)_{k\in\mathcal{N}} \in \prod_{k\in\mathcal{N}}[\underline{c}_k, \infty)$.

Proof. Assume that condition (24) holds, and let $(c_k)_{k\in\mathcal{N}} \in \prod_{k\in\mathcal{N}}[\underline{c}_k, \infty)$. We want to show that condition (22) holds, so let $f \in \mathcal{F}$, $\omega^f \in (0, \bar{\omega}^f)$ and $\mu^f = \frac{1}{1-\omega^f}$. Let $k \in f$ and $p_k = r_k(\mu^f, c_k)$. Since $c_k \geq \underline{c}_k$ and r_k is increasing in its second argument, it follows that $p_k \geq r_k(\mu^f, \underline{c}_k)$. Therefore, $(p_k)_{k\in f} \in \prod_{k\in f}\left[r_k\left(\mu^f, \underline{c}_k\right), \infty\right)$. It follows that condition (22) holds. By Lemma 23, pricing game $\left(\mathcal{N}, (h_k)_{k\in\mathcal{N}}, \mathcal{F}, (c_k)_{k\in\mathcal{N}}\right)$ has a unique equilibrium.

Finally, we show that conditions (24) and (25) are equivalent. Let $f \in \mathcal{F}$, $\omega^f \in (0, \bar{\omega}^f)$, and $(x_k)_{k\in f} \in \prod_{k\in f}\left[r_k\left(\frac{1}{1-\omega^f}, \underline{c}_k\right), \infty\right)$. Then,

$$\begin{aligned}
&\left(\sum_{k\in f}\frac{\omega^f\theta_k}{1-\omega^f\theta_k}\gamma_k\right)\left(\frac{1}{\sum_{k\in f}h_k} - \frac{\omega^f}{\sum_{k\in f}\gamma_k}\right) < 1 \\
\iff &\left(\sum_{i\in f}\frac{\omega^f\theta_i}{1-\omega^f\theta_i}\gamma_i\right)\left(\sum_{j\in f}\left(\gamma_j - \omega^f h_j\right)\right) - \left(\sum_{i\in f}h_i\right)\left(\sum_{j\in f}\gamma_j\right) < 0, \\
\iff &\left(\sum_{i\in f}\frac{\omega^f\theta_i}{1-\omega^f\theta_i}\gamma_i\right)\left(\sum_{j\in f}\gamma_j\left(1-\omega^f\rho_j\right)\right) - \left(\sum_{i\in f}\rho_i\gamma_i\right)\left(\sum_{j\in f}\gamma_j\right) < 0, \\
\iff &\sum_{i,j\in f}\gamma_i\gamma_j\left(\omega^f\theta_i\frac{1-\omega^f\rho_j}{1-\omega^f\theta_i} - \rho_i\right) < 0.
\end{aligned}$$

□

We can now prove Lemma 16:

Proof. Let $(c_k)_{k\in\mathcal{N}} \in \mathbb{R}^{\mathcal{N}}_{++}$, and assume that condition (11) holds. Let $f \in \mathcal{F}$, $\omega^f \in (0, \bar{\omega}^f)$ and $\mu^f = 1/(1-\omega^f)$. Let $(x_k)_{k\in f} \in \prod_{k\in f}\left[r_k(\mu^f, c_k), \infty\right)$. Then, for every $k \in f$,

$$\iota_k(x_k) > \nu_k(x_k, c_k) = \mu^f.$$

Therefore,

$$(x_k)_{k\in f} \in \left\{(x_k)_{k\in f} \in \mathbb{R}^f_{++} :\ \forall k \in f,\ \chi_k(x_k) > \omega^f\right\},$$

and, by condition (11), condition (24) holds for $(\underline{c}_k)_{k\in\mathcal{N}} = (c_k)_{k\in\mathcal{N}}$. By Lemma 24, pricing game $\left(\mathcal{N}, (h_k)_{k\in\mathcal{N}}, \mathcal{F}, (c_k)_{k\in\mathcal{N}}\right)$ has a unique equilibrium. In addition, as shown in the proof of Lemma 24, conditions (11) and (12) are equivalent. □

C.3 Proof of Theorem 4

Proof. Let us show that condition (12) holds for every firm. Let $f \in \mathcal{F}$, $\omega^f \in (0, \bar{\omega}^f)$, and

$$(x_k)_{k\in f} \in \left\{(x_k)_{k\in f} \in \mathbb{R}^f_{++} :\ \forall k \in f,\ \chi_k(x_k) > \omega^f\right\}.$$

Since for every $k \in f$, $\chi_k(x_k) > \omega^f$, it follows that $\iota_k(x_k) > 1$. Therefore, $x_k > \underline{p}_k$ for every k, and, by Assumption 4,

$$\max_{k\in f}\theta_k(x_k) \leq \min_{k\in f}\rho_k(x_k).$$

Therefore,

$$\sum_{i,j\in f} \gamma_i\gamma_j \left(\omega^f\theta_i \frac{1-\omega^f\rho_j}{1-\omega^f\theta_i} - \rho_i\right) \leq \sum_{i,j\in f} \gamma_i\gamma_j \left(\omega^f\rho_i - \rho_i\right),$$
$$\leq \left(\omega^f - 1\right) \sum_{i,j\in f} \gamma_i\gamma_j\rho_i < 0,$$

where the first inequality follows by Lemma 16 and $\max_{k\in f} \theta_k(x_k) \leq \min_{k\in f} \rho_k(x_k)$. By Lemma 17, pricing game $\left(\mathcal{N}, (h_k)_{k\in\mathcal{N}}, \mathcal{F}, (c_k)_{k\in\mathcal{N}}\right)$ has a unique equilibrium for every $(c_k)_{k\in\mathcal{N}} \in \mathbb{R}^{\mathcal{N}}_{++}$. □

C.4 Proof of Proposition 4

Proof. We focus on firm k and drop subscript k from now on. We know that $\lim_\infty h = \lim_\infty \gamma = 0$ and that $\lim_\infty \frac{h'}{\gamma'}$ exists. By l'Hospital's rule,

$$\lim_\infty \rho = \lim_\infty \frac{h}{\gamma} = \lim_\infty \frac{h'}{\gamma'} = \lim_\infty \theta.$$

By Assumption 4, ρ is non-decreasing. This implies that $\lim_\infty \rho = \sup_{(\underline{p},\infty)} \rho$. Therefore,

$$\sup_{(\underline{p},\infty)} \rho \leq \sup_{(\underline{p},\infty)} \theta \leq \inf_{(\underline{p},\infty)} \rho,$$

where the second inequality follows from Assumption 4. Therefore, there exists $K > 0$ such that $\rho(x) = K$ for all $x > \underline{p}$. This is equivalent to:

$$\frac{-h''(x)}{-h'(x)} = K\frac{h'(x)}{h(x)} \ \forall x > \underline{p}.$$

Therefore, there exists $L \in \mathbb{R}$ such that

$$\log\left(-h'(x)\right) = K \log\left(h(x)\right) + L, \ \forall x > \underline{p}.$$

Taking exponentials, we get: $\frac{-h'(x)}{h(x)^K} = \mathrm{e}^L$. We distinguish two cases. Assume first that $K = 1$. Then, there exists $M > 0$ such that $h(x) = M\exp\left(-\mathrm{e}^L x\right)$ for all $x > \underline{p}$, i.e., h is logit on $(\underline{p}, \infty)$.

Next, assume $K \neq 1$. Then, there exists $M \in \mathbb{R}$ such that

$$\frac{h(x)^{1-K}}{K-1} = \mathrm{e}^{L}\, x + M,\ \forall x > \underline{p}.$$

If $K < 1$, then $h(x)^{1-K} < 0$ for x high enough, which is impossible. Therefore, $K > 1$. Similarly, if $M < 0$, then, $h\left(-M\,\mathrm{e}^{-L}\right)^{1-K} = 0$, which cannot be since $h > 0$. Therefore, $M \geq 0$,

$$h(x) = \left((K-1)\,\mathrm{e}^{L}\, x + (K-1)M\right)^{\frac{1}{1-K}},\ \forall x > \underline{p},$$

and h is CES on $(\underline{p}, \infty)$. □

C.5 Proof of Proposition 5

Proof. We focus on product k and drop subscript k from now on. Since $\bar{\mu}^f = 1$, by definition of $\bar{\omega}^f$, $\omega^f = 1$. As in the proof of Proposition 4, it follows from l'Hospital's rule that $\lim_\infty \rho = \lim_\infty \theta$. We have shown in the proof of Lemma 24 that $1 - \omega^f \theta(x) > 0$ if $\omega^f \in (0, \bar{\omega}^f) = (0, 1)$ and $\chi(x) > \omega^f$. Therefore, $\lim_\infty \theta \leq 1/\omega^f$ for all $\omega^f \in (0, 1)$, and $\lim_\infty \theta \leq 1$. It follows that $\lim_\infty \rho \leq 1$. Since $\rho \geq 1$ and ρ is non-decreasing, this implies that $\rho = 1$. Integrating this differential equation as we did in the proof of Proposition 4, we can conclude that ρ is logit. □

C.6 Proof of Lemma 18

Proof. Let $k \in f$ and $\omega^f \in (0, \bar{\omega}^f)$. By Lemma 6, $\lim_\infty \rho_k = \frac{\bar{\mu}^f}{\bar{\mu}^f - 1} = \frac{1}{\bar{\omega}^f}$. By Assumption 3, ρ_k is non-decreasing. Therefore, $\rho_k(x) \leq \frac{1}{\bar{\omega}^f}$ for all $x > \underline{p}_k$. In particular, this inequality is also satisfied if x is such that $\chi_k(x) > \omega^f$.

In addition, as shown in the proof of Lemma 6,

$$\rho_k(x) = \iota_k(x) \frac{h_k(x)}{-x h_k'(x)}.$$

Therefore,

$$\begin{aligned}
\frac{d \log \rho_k(x)}{dx} &= \frac{\iota_k'(x)}{\iota_k(x)} + \left(\frac{h_k'(x)}{h_k(x)} - \frac{1}{x} + \frac{h_k''(x)}{-h_k'(x)}\right), \\
&= \frac{\iota_k'(x)}{\iota_k(x)} + \frac{1}{x}\left(-\frac{\iota_k(x)}{\rho_k(x)} - 1 + \iota_k(x)\right), \\
&= \frac{\iota_k'(x)}{\iota_k(x)} + \frac{\iota_k(x)}{x \rho_k(x)} \left(\rho_k(x) \chi_k(x) - 1\right),
\end{aligned}$$

$$\leq \frac{\iota_k'(x)}{\iota_k(x)},$$

where the last inequality follows from the fact that $\chi_k(x) \leq \bar{\omega}^f$ and $\rho_k(x) \leq \frac{1}{\bar{\omega}^f}$. Therefore, for all $x > \underline{p}_k$,

$$\begin{aligned}
\log\left(\frac{1}{\bar{\omega}^f \rho_k(x)}\right) &= \int_x^\infty \frac{\rho_k'(t)}{\rho_k(t)} dt, \\
&\leq \int_x^\infty \frac{\iota_k'(t)}{\iota_k(t)} dt, \\
&= \log\left(\frac{\bar{\mu}^f}{\iota_k(x)}\right), \\
&= \log\left(\frac{1-\chi_k(x)}{1-\bar{\omega}^f}\right).
\end{aligned}$$

Therefore,

$$\rho_k(x) \geq \frac{1-\bar{\omega}^f}{\bar{\omega}^f} \frac{1}{1-\chi_k(x)}, \ \forall x > \underline{p}_k.$$

In particular, if $\chi_k(x) > \omega^f$, then

$$\rho_k(x) \geq \frac{1-\bar{\omega}^f}{\bar{\omega}^f} \frac{1}{1-\omega^f}.$$

□

C.7 Proof of Lemma 19

Proof. Let $\bar{\omega} \in (0,1)$ and $\omega \in (0, \bar{\omega})$. Define

$$M(\omega, \bar{\omega}) = \max_{(y,z) \in \left[\frac{1-\bar{\omega}}{\bar{\omega}} \frac{1}{1-\omega}, \frac{1}{\bar{\omega}}\right]^2} \phi_{\omega,\bar{\omega}}(y,z).$$

Notice that $\phi_{\omega,\bar{\omega}}(y,z) = \phi_{\omega,\bar{\omega}}(z,y)$ for every y and z. It follows that

$$M(\omega, \bar{\omega}) = \max_{\substack{(y,z) \in \left[\frac{1-\bar{\omega}}{\bar{\omega}} \frac{1}{1-\omega}, \frac{1}{\bar{\omega}}\right]^2 \\ y \leq z}} \phi_{\omega,\bar{\omega}}(y,z).$$

Let $\frac{1-\bar{\omega}}{\bar{\omega}} \frac{1}{1-\omega} \leq y \leq z \leq \frac{1}{\bar{\omega}}$. Then,

$$\frac{\partial \phi_{\omega,\bar{\omega}}}{\partial y} = \frac{\omega(1-\omega z)}{(1-\omega y)^2} - \frac{\omega^2 z}{1-\omega z} - 1,$$

$$
\begin{aligned}
&= \frac{1}{1-\omega z}\left(\omega\left(\frac{1-\omega z}{1-\omega y}\right)^2 - \omega^2 z - (1-\omega z)\right), \\
&\leq \frac{1}{1-\omega z}\left(\omega - \omega^2 z - (1-\omega z)\right), \text{ since } y \leq z, \\
&= \omega - 1 < 0.
\end{aligned}
$$

It follows that, for every $(y,z) \in \left[\frac{1-\bar{\omega}}{\bar{\omega}}\frac{1}{1-\omega}, \frac{1}{\bar{\omega}}\right]^2$ such that $y \leq z$,

$$
\phi_\omega(y,z) \leq \phi_\omega\left(\frac{1-\bar{\omega}}{\bar{\omega}}\frac{1}{1-\omega}, z\right) \equiv \psi_{\omega,\bar{\omega}}(z).
$$

Therefore,

$$
M(\omega,\bar{\omega}) = \max_{z\in\left[\frac{1-\bar{\omega}}{\bar{\omega}}\frac{1}{1-\omega},\frac{1}{\bar{\omega}}\right]} \psi_{\omega,\bar{\omega}}(z).
$$

Since

$$
\psi''_{\omega,\bar{\omega}}(z) = \left(1 - \frac{\omega}{1-\omega}\frac{1-\bar{\omega}}{\bar{\omega}}\right)\frac{2\omega^2}{(1-\omega z)^3} > 0,
$$

function $\psi_{\omega,\bar{\omega}}(.)$ is strictly convex. Therefore,

$$
M(\omega,\bar{\omega}) = \max\left\{\phi_{\omega,\bar{\omega}}\left(\frac{1-\bar{\omega}}{\bar{\omega}}\frac{1}{1-\omega}, \frac{1-\bar{\omega}}{\bar{\omega}}\frac{1}{1-\omega}\right), \phi_{\omega,\bar{\omega}}\left(\frac{1-\bar{\omega}}{\bar{\omega}}\frac{1}{1-\omega}, \frac{1}{\bar{\omega}}\right)\right\}.
$$

Since $\phi_{\omega,\bar{\omega}}(z,z) = 2(\omega-1)z < 0$ for every z, it follows that $M(\omega,\bar{\omega}) \leq 0$ if and only if $\zeta(\omega,\bar{\omega}) \leq 0$, where

$$
\begin{aligned}
\zeta(\omega,\bar{\omega}) &\equiv \phi\left(\frac{1-\bar{\omega}}{\bar{\omega}}\frac{1}{1-\omega}, \frac{1}{\bar{\omega}}\right), \\
&= \left(1-\frac{\omega}{\bar{\omega}}\right)\frac{\frac{\omega}{1-\omega}\frac{1-\bar{\omega}}{\bar{\omega}}}{1-\frac{\omega}{1-\omega}\frac{1-\bar{\omega}}{\bar{\omega}}} + \frac{\omega}{\bar{\omega}-\omega}\left(1-\frac{\omega}{1-\omega}\frac{1-\bar{\omega}}{\bar{\omega}}\right) - \frac{1-\bar{\omega}}{\bar{\omega}}\frac{1}{1-\omega} - \frac{1}{\bar{\omega}}, \\
&= \frac{\omega(1-\bar{\omega})}{\bar{\omega}} + \frac{\omega}{(1-\omega)\bar{\omega}} - \frac{1-\bar{\omega}}{\bar{\omega}}\frac{1}{1-\omega} - \frac{1}{\bar{\omega}}, \\
&= \frac{1}{1-\omega} + \frac{\omega-2}{\bar{\omega}} - \omega.
\end{aligned}
$$

For every $\omega \in (0,\bar{\omega})$,

$$
\frac{\partial\zeta}{\partial\omega} = \frac{1}{(1-\omega)^2} + \frac{1}{\bar{\omega}} - 1 > 0.
$$

Therefore, ζ is strictly increasing in ω on interval $(0, \bar{\omega})$. It follows that $M(\omega, \bar{\omega}) \leq 0$ for every $\omega \in (0, \bar{\omega})$ if and only if $\xi(\bar{\omega}) \leq 0$, where

$$\begin{aligned}\xi(\bar{\omega}) &\equiv \zeta(\bar{\omega}, \bar{\omega}), \\ &= \frac{1}{1-\bar{\omega}} + 1 - \bar{\omega} - \frac{2}{\bar{\omega}}.\end{aligned}$$

For every $\bar{\omega} \in (0,1)$,

$$\xi'(\bar{\omega}) = \frac{1}{(1-\bar{\omega})^2} + \frac{2}{(\bar{\omega})^2} - 1 > 0.$$

Therefore, ξ is strictly increasing on $(0,1)$. Since $\lim_{0^+} \xi = -\infty$ and $\lim_{1^-} = +\infty$, there exists a unique threshold $\omega^* \in (0,1)$ such that $\xi(\bar{\omega}) \leq 0$ if and only if $\bar{\omega} \leq \omega^*$. Numerically, we find that $\omega^* \simeq 0.64$. This concludes the proof. □

C.8 Proof of Theorem 5

Proof. Assume that $\bar{\omega}^f < \omega^*$. Splitting the sum in two terms, condition (12) can be rewritten as follows:

$$\begin{aligned}&\forall \omega^f \in (0, \bar{\omega}^f),\ \forall (x_k)_{k\in f} \in \left\{(x_k)_{k\in f} \in \mathbb{R}^f_{++} :\ \forall k \in f,\ \chi_k(x_k) > \omega^f\right\}, \\ &\frac{1}{2}\sum_{\substack{i,j\in f \\ i\neq j}} \gamma_i(x_i)\gamma_j(x_j)\left(\omega^f\theta_i(x_i)\frac{1-\omega^f\rho_j(x_j)}{1-\omega^f\theta_i(x_i)} + \omega^f\theta_j(x_j)\frac{1-\omega^f\rho_i(x_i)}{1-\omega^f\theta_j(x_j)} - \rho_i(x_i) - \rho_j(x_j)\right) \\ &+\left(\sum_{i\in f}\gamma_i(x_i)^2\left(\omega^f\theta_i(x_i)\frac{1-\omega^f\rho_i(x_i)}{1-\omega^f\theta_i(x_i)} - \rho_i(x_i)\right)\right) < 0.\end{aligned} \tag{26}$$

Let us first show that the second sum is strictly negative. Let $\omega^f \in (0, \bar{\omega}^f)$, $i \in f$ and x_i such that $\chi_i(x_i) > \omega^f$. Therefore,

$$\omega^f\theta_i(x_i)\frac{1-\omega^f\rho_i(x_i)}{1-\omega^f\theta_i(x_i)} - \rho_i(x_i) \leq \omega^f\theta_i(x_i) - \rho_i(x_i) < 0,$$

where the first inequality follows from Assumption 3 ($\theta_i(x_i) \leq \rho_i(x_i)$) and Lemma 16 ($1 - \omega^f\theta_i(x_i) > 0$).

Next, we turn our attention to the first sum. Let $\omega^f \in (0, \bar{\omega}^f)$ and $(x_k)_{k\in f}$ such that $\chi_k(x_k) > \omega^f$ for every $k \in f$. By Lemma 18,

$$\forall k \in f,\ \rho_k(x) \in \left[\frac{1-\bar{\omega}^f}{\bar{\omega}^f}\frac{1}{1-\omega^f}, \frac{1}{\bar{\omega}^f}\right].$$

In addition, as shown above, for every $k \in f$, $\theta_k(x_k) \leq \rho_k(x_k) \left(< \frac{1}{\omega^f}\right)$. Therefore,

$$\begin{aligned}
&\frac{1}{2}\sum_{\substack{i,j\in f\\ i\neq j}} \gamma_i(x_i)\gamma_j(x_j)\left(\omega^f\theta_i(x_i)\frac{1-\omega^f\rho_j(x_j)}{1-\omega^f\theta_i(x_i)} + \omega^f\theta_j(x_j)\frac{1-\omega^f\rho_i(x_i)}{1-\omega^f\theta_j(x_j)} - \rho_i(x_i) - \rho_j(x_j)\right)\\
&\leq \frac{1}{2}\sum_{\substack{i,j\in f\\ i\neq j}} \gamma_i(x_i)\gamma_j(x_j)\phi_{\omega^f,\bar{\omega}^f}\left(\rho_i(x_i),\rho_j(x_j)\right),\\
&\leq 0, \text{ by Lemma 19.}
\end{aligned}$$

This concludes the proof. □

C.9 Uniqueness when $\bar{\mu}^f$ is Small

In this section, we extend Theorem 5 to cases where $\lim_\infty h_k$ is not necessarily equal to zero. We start with the following technical lemma:

Lemma 25. *Let $\left(\mathcal{N}, (h_k)_{k\in\mathcal{N}}, \mathcal{F}\right)$ be a pricing structure satisfying Assumptions 1 and 2. Let $f \in \mathcal{F}$. Assume that Assumption 3 is satisfied for firm f. Then, for every $k \in f$,*

$$S_k = \left\{\omega \in (0,\bar{\omega}^f): \ \exists x > \underline{p}_k,\ \omega = \chi_k(x) = \frac{1}{\rho_k(x)}\right\}$$

contains at most one element. If S_k is empty, then, either $\chi_k(x)\rho_k(x) > 1$ for every $x > \underline{p}_k$, or $\chi_k(x)\rho_k(x) < 1$ for every $x > \underline{p}_k$. If, instead, $S_k = \{\hat{\omega}\}$, then, for every $x > \underline{p}_k$,

- *$\theta_k(x) \leq \frac{1}{\hat{\omega}}$, and*
- *if $\rho_k(x) < \frac{1}{\hat{\omega}}$, then $\rho_k(x) \geq \frac{1-\hat{\omega}}{\hat{\omega}}\frac{1}{1-\chi_k(x)}$.*

Proof. Let $k \in f$, and assume for a contradiction that S_k contains two distinct elements. There exist $x, y > \underline{p}_k$ such that $\chi_k(x)\rho_k(x) = 1$, $\chi_k(y)\rho_k(y) = 1$ and $\chi_k(x) \neq \chi_k(y)$. To fix ideas, assume $\chi_k(y) > \chi_k(x)$. Then, since χ_k is non-decreasing, $y > x$. Since ρ_k is non-decreasing, $\rho_k(x) \leq \rho_k(y)$. Therefore, $\chi_k(x)\rho_k(x) < \chi_k(y)\rho_k(y) = 1$, which is a contradiction.

Let $\kappa : x \in (\underline{p}_k, \infty) \mapsto \rho_k(x)\chi_k(x)$, and notice that κ is continuous and non-decreasing. If $S_k = \emptyset$, then, there is no x such that $\kappa(x) = 1$. Since κ is continuous, either $\kappa > 1$, or $\kappa < 1$.

Next, let $x > \underline{p}_k$. If $\rho_k(x) \leq \frac{1}{\hat{\omega}}$, then, $\theta_k(x) \leq \rho_k(x) \leq \frac{1}{\hat{\omega}}$. Assume instead that $\rho_k(x) > \frac{1}{\hat{\omega}}$. Let $\hat{x}$ such that $\chi_k(\hat{x}) = \hat{\omega} = \frac{1}{\rho_k(\hat{x})}$. Then, $\rho_k(x) > \rho_k(\hat{x}) = \frac{1}{\hat{\omega}}$ and, by monotonicity, $x > \hat{x}$. Therefore, $\chi_k(x) \geq \chi_k(\hat{x}) = \hat{\omega}$. Next, we claim that $\theta_k(x) \leq \frac{1}{\chi_k(x)}$. To see this, notice that

$\iota_k(x) = x\frac{-h_k'(x)}{\gamma_k(x)}$. Therefore,

$$\begin{aligned}\frac{\iota_k'(x)}{\iota_k(x)} &= \frac{1}{x} + \frac{h_k''(x)}{h_k'(x)} - \frac{\gamma_k'(x)}{\gamma_k(x)},\\ &= \frac{1}{x}\left(1 - \iota_k(x) + \frac{\gamma_k'(x)}{h_k'(x)} x \frac{-h_k'(x)}{\gamma_k(x)}\right),\\ &= \frac{1}{x}\left(1 - \iota_k(x) + \frac{\iota_k(x)}{\theta_k(x)}\right).\end{aligned}$$

Therefore,

$$\theta_k(x) = \frac{\iota_k(x)}{\iota_k(x) - 1 + x\frac{\iota_k'(x)}{\iota_k(x)}} \leq \frac{\iota_k(x)}{\iota_k(x) - 1} = \frac{1}{\chi_k(x)}.$$

Therefore, $\theta_k(x) \leq \frac{1}{\chi_k(x)} \leq \frac{1}{\hat{\omega}}$.

Next, assume that $\rho_k(x) < \frac{1}{\hat{\omega}}$. We know from the proof of Lemma 18 that for every $t \in [x, \hat{x}]$,

$$\begin{aligned}\frac{\rho_k'(t)}{\rho_k(t)} &= \frac{\iota_k'(t)}{\iota_k(t)} + \frac{\iota_k(t)}{t\rho_k(t)}\left(\rho_k(t)\chi_k(t) - 1\right),\\ &\leq \frac{\iota_k'(t)}{\iota_k(t)} + \frac{\iota_k(t)}{t\rho_k(t)}\left(\rho_k(\hat{x})\chi_k(\hat{x}) - 1\right),\ \text{by monotonicity},\\ &= \frac{\iota_k'(t)}{\iota_k(t)},\ \text{since } \rho_k(\hat{x})\chi_k(\hat{x}) = 1.\end{aligned}$$

Integrating this inequality between x and $\hat{x}$, we obtain that $\frac{\rho_k(\hat{x})}{\rho_k(x)} \leq \frac{\iota_k(\hat{x})}{\iota_k(x)}$. Therefore,

$$\begin{aligned}\rho_k(x) &\geq \rho_k(\hat{x})\frac{\iota_k(x)}{\iota_k(\hat{x})},\\ &= \frac{1-\hat{\omega}}{\hat{\omega}}\frac{1}{1-\chi_k(x)}.\end{aligned}$$

□

Proposition 15. *Let* $\left(\mathcal{N}, (h_k)_{k\in\mathcal{N}}, \mathcal{F}\right)$ *be a pricing structure satisfying Assumptions 1 and 2. Let* $f \in \mathcal{F}$*. Assume that Assumption 3 is satisfied for firm* f *and that* $\bar{\omega}^f \leq \omega^*$*. Assume also, using the notations introduced in Lemma 25 that, for every* $i \in f$*,* $S_i = \{\hat{\omega}\}$ *. Then, condition* (12) *holds for firm* f*.*

Proof. As in the proof of Theorem 5, the expression in condition (12) can be split in two terms (see equation (26)). Given Assumption 3 and Lemma 16, the second sum is strictly negative. Next, we turn our attention to the first sum. Let $\omega^f \in (0, \bar{\omega}^f)$, $i, j \in f$, and x_i, x_j

such that $\chi_i(x_i) > \omega^f$ and $\chi_j(x_j) > \omega^f$. We want to show that

$$\Psi = \omega^f \theta_i(x_i) \frac{1 - \omega^f \rho_j(x_j)}{1 - \omega^f \theta_i(x_i)} + \omega^f \theta_j(x_j) \frac{1 - \omega^f \rho_i(x_i)}{1 - \omega^f \theta_j(x_j)} - \rho_i(x_i) - \rho_j(x_j) \leq 0. \tag{27}$$

To fix ideas, assume that $\rho_i(x_i) \leq \rho_j(x_j)$. If $\rho_i(x_i) \geq \frac{1}{\omega^f}$, then condition (27) is clearly satisfied, since, by Lemma 16, $1 - \omega^f \theta_i(x_i)$ and $1 - \omega^f \theta_j(x_i)$ are strictly positive. Assume instead that $\rho_i(x_i) < \frac{1}{\omega^f}$. Then, we claim that $\omega^f < \hat{\omega}$. Assume for a contradiction that $\hat{\omega} \leq \omega^f$. Since $S_i = \{\hat{\omega}\}$, there exists $\hat{x}_i > \underline{p}_i$ such that $\chi_i(\hat{x}_i) = \hat{\omega} = \frac{1}{\rho_i(\hat{x}_i)}$. Therefore, $\rho_i(x_i) < \rho_i(\hat{x}_i)$ and, by monotonicity, $x_i < \hat{x}_i$. Since χ_i is non-decreasing, it follows that

$$\omega^f < \chi_i(x_i) \leq \chi_i(\hat{x}_i) = \hat{\omega},$$

which is a contradiction. Therefore, $\omega^f < \hat{\omega}$.

We distinguish three cases. Assume first that $\rho_j(x_j) < \frac{1}{\hat{\omega}}$. Then, by Lemma 25,

$$\rho_k(x_k) \geq \frac{1 - \hat{\omega}}{\hat{\omega}} \frac{1}{1 - \chi_k(x_k)} \geq \frac{1 - \hat{\omega}}{\hat{\omega}} \frac{1}{1 - \omega^f},$$

for $k \in \{i, j\}$. In addition, $\frac{\theta_i(x_i)}{1-\omega^f \theta_i(x_i)} \leq \frac{\rho_i(x_i)}{1-\omega^f \rho_i(x_i)}$ and $\frac{\theta_j(x_j)}{1-\omega^f \theta_j(x_j)} \leq \frac{\rho_j(x_j)}{1-\omega^f \rho_j(x_j)}$. Therefore,

$$\Psi \leq \phi_{\omega^f, \hat{\omega}} \left(\rho_i(x_i), \rho_j(x_j) \right),$$

which, by Lemma 19, is non-positive, since $\hat{\omega} < \bar{\omega}^f \leq \omega^*$.

Next, assume that $\rho_i(x_i) < \frac{1}{\hat{\omega}} \leq \rho_j(x_j)$. Then, by Lemma 25,

$$\rho_i(x_i) \geq \frac{1 - \hat{\omega}}{\hat{\omega}} \frac{1}{1 - \chi_i(x_i)} \geq \frac{1 - \hat{\omega}}{\hat{\omega}} \frac{1}{1 - \omega^f},$$

and $\theta_j(x_j) \leq \frac{1}{\hat{\omega}}$. Therefore,

$$\begin{aligned}
\Psi &\leq \frac{\omega^f \theta_i(x_i)}{1 - \omega^f \theta_i(x_i)} \left(1 - \frac{\omega^f}{\hat{\omega}} \right) + \frac{\frac{\omega^f}{\hat{\omega}}}{1 - \frac{\omega^f}{\hat{\omega}}} (1 - \omega^f \rho_i(x_i)) - \rho_i(x_i) - \frac{1}{\hat{\omega}}, \\
&\leq \frac{\omega^f \rho_i(x_i)}{1 - \omega^f \rho_i(x_i)} \left(1 - \frac{\omega^f}{\hat{\omega}} \right) + \frac{\frac{\omega^f}{\hat{\omega}}}{1 - \frac{\omega^f}{\hat{\omega}}} (1 - \omega^f \rho_i(x_i)) - \rho_i(x_i) - \frac{1}{\hat{\omega}}, \\
&= \phi_{\omega^f, \hat{\omega}} \left(\rho_i(x_i), \frac{1}{\hat{\omega}} \right), \\
&\leq 0 \text{ by Lemma 19.}
\end{aligned}$$

Finally, assume that $\rho_i(x_i) \geq \frac{1}{\hat{\omega}}$. By Lemma 25, $\theta_i(x_i) \leq \frac{1}{\hat{\omega}}$ and $\theta_j(x_j) \leq \frac{1}{\hat{\omega}}$. Therefore,

$$\begin{aligned}
\Psi &\leq \frac{\omega^f \theta_i(x_i)}{1-\omega^f \theta_i(x_i)} \left(1 - \frac{\omega^f}{\hat{\omega}}\right) + \frac{\omega^f \theta_j(x_j)}{1-\omega^f \theta_j(x_j)} \left(1 - \frac{\omega^f}{\hat{\omega}}\right) - \frac{1}{\hat{\omega}} - \frac{1}{\hat{\omega}}, \\
&\leq \frac{\frac{\omega^f}{\hat{\omega}}}{1-\frac{\omega^f}{\hat{\omega}}} \left(1 - \frac{\omega^f}{\hat{\omega}}\right) + \frac{\frac{\omega^f}{\hat{\omega}}}{1-\frac{\omega^f}{\hat{\omega}}} \left(1 - \frac{\omega^f}{\hat{\omega}}\right) - \frac{1}{\hat{\omega}} - \frac{1}{\hat{\omega}}, \\
&= \phi_{\omega^f,\hat{\omega}} \left(\frac{1}{\hat{\omega}}, \frac{1}{\hat{\omega}}\right), \\
&\leq 0 \text{ by Lemma 19.}
\end{aligned}$$

This concludes the proof. □

Condition $S_i = \{\hat{\omega}\}\ \forall i$ in Proposition 15 may look a little bit arcane. The following corollary is easier to understand:

Corollary 1. *Let $\left(\mathcal{N}, (h_k)_{k\in\mathcal{N}}, \mathcal{F}\right)$ be a pricing structure satisfying Assumptions 1 and 2. Let $f \in \mathcal{F}$. Assume that Assumption 3 is satisfied for firm f and that $\bar{\omega}^f \leq \omega^*$. Assume also that there exist $h \in \mathbb{R}_{++}^{\mathbb{R}_{++}}$ and $(\alpha_k, \beta_k)_{k\in f} \in \left(\mathbb{R}^2_{++}\right)^f$ such that for every $k \in f$, for every $x > 0$, $h_k(x) = \alpha_k h(\beta_k x)$. Then, condition* (12) *holds for firm f.*

Proof. Let us first show that $S_i \subseteq S_j$ for all $i, j \in f$. Let $i, j \in f$. If S_i is empty, then, trivially, $S_i \subseteq S_j$. Assume instead that $S_i \neq \emptyset$, and let $\hat{\omega} \in S_i$. There exists $\hat{x}_i > \underline{p}_i$ such that

$$\chi_i(\hat{x}_i) = \hat{\omega} = \frac{1}{\rho_i(\hat{x}_i)}.$$

Since $h_i(x_i) = \alpha_i h(\beta_i x)$, we also know from the proof of Proposition 6 that $\rho_i(\hat{x}_i) = \rho(\beta_i \hat{x}_i)$ and $\chi_i(\hat{x}_i) = \chi(\beta_i \hat{x}_i)$. Let $\hat{x}_j = \frac{\beta_i}{\beta_j}\hat{x}_i$. Then,

$$\chi_j\left(\hat{x}_j\right) = \chi\left(\beta_j \frac{\beta_i}{\beta_j}\hat{x}_i\right) = \chi_i(\hat{x}_i) = \hat{\omega} = \frac{1}{\rho_i(\hat{x}_i)} = \frac{1}{\rho(\beta_i \hat{x}_i)} = \frac{1}{\rho_j(\hat{x}_j)}.$$

Therefore, $\hat{\omega} \in S_j$, and $S_i \subseteq S_j$. It follows that $S_i = S_j$ for all $i, j \in f$.

If $S_i \neq \emptyset$, then, by Proposition 15, condition (12) holds for firm f. Assume instead that $S_i = \emptyset$ for all i. Let $i \in f$. By Lemma 25, either $\chi_i(x_i)\rho_i(x_i) < 1$ for all x_i, or $\chi_i(x_i)\rho_i(x_i) > 1$ for all x_i. Assume first that $\chi_i(x_i)\rho_i(x_i) < 1$ for all x_i. Let $j \in f$ and $x_j > \underline{p}_j$. Then,

$$\chi_j(x_j)\rho_j(x_j) = \chi_i\left(\frac{\beta_j}{\beta_i}x_j\right)\rho_i\left(\frac{\beta_j}{\beta_i}x_j\right) < 1.$$

Therefore, $\chi_j \rho_j < 1$ for every j in f. It follows that

$$\lim_{\infty} \rho_j \leq \lim_{\infty} \frac{1}{\chi_j} = \frac{1}{\bar{\omega}^f} < \infty.$$

Therefore, $\lim_\infty h_j = 0$ for every $j \in f$ (if $\lim_\infty h_j$ were strictly positive, then $\rho_j(x_j)$ would go to ∞ as x_j goes to ∞). By Theorem 5, condition (12) holds for firm f.

Finally, assume that $\chi_i(x_i)\rho_i(x_i) > 1$ for all x_i. Then, using the same argument as above, $\chi_j \rho_j > 1$ for every $j \in f$. Let $i \in f$, and assume for a contradiction that $\underline{p}_i > 0$. Since $1/\chi_i$ is non-increasing, and since, by continuity, $\iota_i(\underline{p}_i) = 1$, it follows that $\lim_{\underline{p}_i^+} \frac{1}{\chi_i} = \infty$. Therefore, $\lim_{\underline{p}_i^+} \rho_i = \infty$, which is a contradiction, since ρ_i is non-decreasing. Therefore, $\underline{p}_i = 0$.

Assume for a contradiction that $\lim_{0^+} \iota_i = 1$. Then, using the same reasoning as in the previous paragraph, $\lim_{0^+} \rho_i = \infty$, which is again a contradiction, since ρ_i is non-decreasing. Therefore, $\lim_{0^+} \iota_i > 1$, and $\hat{\omega} \equiv \lim_{0^+} \chi_i$ is strictly positive. In addition, since

$$\chi_j(x) = \chi_i \left(\frac{\beta_j}{\beta_i} x \right),$$

$\lim_{0^+} \chi_j = \hat{\omega}$ for every $j \in f$. Notice that, for every $j \in f$, for every $x > 0$,

$$\rho_j(x) \geq \lim_{0^+} \rho_j \geq \lim_{0^+} \frac{1}{\chi_j} = \frac{1}{\hat{\omega}},$$

and that, by Lemma 16,

$$\theta_j(x) \leq \frac{1}{\chi_j(x)} \leq \lim_{0^+} \frac{1}{\chi_j} = \frac{1}{\hat{\omega}}.$$

It follows that

$$\max_{i \in f} \sup \theta_i \leq \frac{1}{\hat{\omega}} \leq \min_{i \in f} \inf \rho_i,$$

i.e., Assumption 4 holds. As shown in the proof of Theorem 4, condition (12) is therefore satisfied for firm f. □

Proposition 16. *Let $\left(\mathcal{N}, (h_k)_{k \in \mathcal{N}}, \mathcal{F}\right)$ be a pricing structure satisfying Assumptions 1 and 2. Let $f \in \mathcal{F}$. Assume that Assumption 3 is satisfied for firm f, that $\bar{\omega}^f \leq \omega^*$, and that $\theta_k \leq \frac{1}{\bar{\omega}^f}$ for every k in f. Then, condition* (12) *holds for firm f.*

Proof. Let $i, j \in f$, $\omega^f \in (0, \bar{\omega}^f)$ and $x_i, x_j > 0$ such that $\chi_i(x_i) > \omega^f$ and $\chi_j(x_j) > \omega^f$. Define

$$\Psi = \frac{\omega^f \theta_i(x_i)}{1 - \omega^f \theta_i(x_i)} \left(1 - \omega^f \rho_j(x_j)\right) + \frac{\omega^f \theta_j(x_j)}{1 - \omega^f \theta_j(x_j)} \left(1 - \omega^f \rho_i(x_i)\right) - \rho_i(x_i) - \rho_j(x_j).$$

As in the previous proofs, all we need to do is show that $\Psi \leq 0$. Assume first that $\rho_i(x_i) \geq \frac{1}{\bar{\omega}^f}$ and $\rho_j(x_j) \geq \frac{1}{\bar{\omega}^f}$. Then,

$$\max\left(\theta_i(x_i), \theta_j(x_j)\right) \leq \min\left(\rho_i(x_i), \rho_j(x_j)\right).$$

Therefore, $\Psi < 0$.

Next, assume that $\rho_i(x_i) < \frac{1}{\bar{\omega}^f}$ and $\rho_j(x_j) \geq \frac{1}{\bar{\omega}^f}$. Then, we claim that

$$\rho_i(x_i) \geq \frac{1-\bar{\omega}^f}{\bar{\omega}^f}\frac{1}{1-\omega^f}. \tag{28}$$

To see this, assume first that $S_i = \{\hat{\omega}_i\}$, where $\hat{\omega}_i \in (0, \bar{\omega}^f)$. Since $\rho_i(x_i) < \frac{1}{\bar{\omega}^f} < \frac{1}{\hat{\omega}^i}$, by Lemma 25,

$$\rho_i(x_i) \geq \frac{1-\hat{\omega}_i}{\hat{\omega}_i}\frac{1}{1-\chi_i(x_i)} \geq \frac{1-\bar{\omega}^f}{\bar{\omega}^f}\frac{1}{1-\omega^f}.$$

Assume instead that $S_i = \emptyset$. By Lemma 25, either $\chi_i\rho_i < 1$ or $\chi_i\rho_i > 1$. If $\chi_i\rho_i > 1$, then we know from the proof of Corollary 1 that

$$\rho_i \geq \sup\frac{1}{\chi_i} \geq \frac{1}{\bar{\omega}^f}.$$

This contradicts our assumption that $\rho_i(x_i) < \frac{1}{\bar{\omega}^f}$. If, instead, $\chi_i\rho_i < 1$, then we know from the proof of Corollary 1 that $\lim_\infty h_i = 0$. Therefore, by Lemma 18, inequality (28) holds.

Therefore,

$$\begin{aligned}
\Psi &\leq \frac{\omega^f\theta_i(x_i)}{1-\omega^f\theta_i(x_i)}\left(1-\frac{\omega^f}{\bar{\omega}^f}\right) + \frac{\frac{\omega^f}{\bar{\omega}^f}}{1-\frac{\omega^f}{\bar{\omega}^f}}\left(1-\omega^f\rho_i(x_i)\right) - \rho_i(x_i) - \frac{1}{\bar{\omega}^f},\\
&\leq \frac{\omega^f\rho_i(x_i)}{1-\omega^f\rho_i(x_i)}\left(1-\frac{\omega^f}{\bar{\omega}^f}\right) + \frac{\frac{\omega^f}{\bar{\omega}^f}}{1-\frac{\omega^f}{\bar{\omega}^f}}\left(1-\omega^f\rho_i(x_i)\right) - \rho_i(x_i) - \frac{1}{\bar{\omega}^f},\\
&= \phi_{\omega^f,\bar{\omega}^f}\left(\rho_i(x_i), \frac{1}{\bar{\omega}^f}\right),\\
&\leq 0 \text{ by Lemma 19.}
\end{aligned}$$

Finally, assume that $\rho_i(x_i) < \frac{1}{\bar{\omega}^f}$ and $\rho_j(x_j) < \frac{1}{\bar{\omega}^f}$. Then, as above,

$$\rho_k(x_k) \geq \frac{1-\bar{\omega}^f}{\bar{\omega}^f}\frac{1}{1-\omega^f}$$

for $k \in \{i, j\}$. Therefore,

$$\Psi \leq \phi_{\omega^f, \bar{\omega}^f}\left(\rho_i(x_i), \rho_j(x_j)\right),$$

which is non-positive by Lemma 19. □

Corollary 2. *Let $\left(\mathcal{N}, (h_k)_{k \in \mathcal{N}}, \mathcal{F}\right)$ be a pricing structure satisfying Assumptions 1 and 2. Let $f \in \mathcal{F}$. Assume that Assumption 3 is satisfied for firm f, that $\bar{\omega}^f \leq \omega^*$, and that θ_k is non-decreasing for every k in f. Then, condition* (12) *holds for firm f.*

Proof. Let $k \in f$. Since θ_k is non-increasing, for every $x > \underline{p}_k$,

$$\theta_k(x) \leq \sup \theta_k = \lim_{\infty} \theta_k \leq \lim_{\infty} \frac{1}{\chi_k} = \frac{1}{\bar{\omega}^f},$$

where the second inequality follows from Lemma 16. Therefore, by Proposition 16, condition (12) holds for firm f. □

C.10 Other Uniqueness Results

Proposition 17. *Let $\left(\mathcal{N}, (h_k)_{k \in \mathcal{N}}, \mathcal{F}, (c_k)_{k \in \mathcal{N}}\right)$ be a pricing game satisfying Assumptions 1 and 2. Let $f \in \mathcal{F}$. Assume that Assumption 3 is satisfied for firm f. Assume also that there exist $h \in \mathbb{R}_{++}^{\mathbb{R}_{++}}$, $c > 0$ and $(\alpha_k)_{k \in f} \in \mathbb{R}_{++}^f$ such that for every k in f, $c_k = c$, and for every $x > 0$, $h_k(x) = \alpha_k h(x)$. Then, condition* (22) *holds.*

Proof. Let $k \in f$. It follows immediately from the proof of Proposition 6 that $\theta_k = \theta$, $\rho_k = \rho$, $\gamma_k = \alpha_k \gamma$, $\iota_k = \iota$, and $\chi_k = \chi$. In addition, $\nu_k = \nu$. Therefore, $r_k = r$. Condition (22) is equivalent to

$$\forall \omega^f \in (0, \bar{\omega}^f), \left(\sum_{k \in f} \frac{\omega^f \theta}{1 - \omega^f \theta} \alpha_k \gamma\right) \left(\frac{1}{\sum_{k \in f} \alpha_k h} - \frac{\omega^f}{\sum_{k \in f} \alpha_k \gamma}\right) < 1,$$

where all functions are evaluated at $r\left(\frac{1}{1-\omega^f}\right)$. This is equivalent to

$$\frac{1 - \omega^f \rho}{1 - \omega^f \theta} \frac{\omega^f \theta}{\rho} < 1,$$

which clearly holds, since $\theta \leq \rho$. □

Corollary 3. *Let $\left(\mathcal{N}, (h_k)_{k \in \mathcal{N}}, \mathcal{F}, (c_k)_{k \in \mathcal{N}}\right)$ be a pricing game satisfying Assumptions 1 and 2. Let $f \in \mathcal{F}$. Assume that $|f| = 1$ and that assumption 3 is satisfied for firm f. Then, condition* (22) *holds.*

Proof. This follows immediately from Proposition 17. □

Proposition 18. *Let* $\left(\mathcal{N}, (h_k)_{k\in\mathcal{N}}, \mathcal{F}\right)$ *be a pricing structure satisfying Assumptions 1 and 2. Assume that there exists* $k \in \mathcal{N}$ *such that* $\lim_\infty h_k > 0$. *Then, there exists* $(\underline{c}_k)_{k\in\mathcal{N}} \in \mathbb{R}^{\mathcal{N}}_{++}$ *such that pricing game* $\left(\mathcal{N}, (h_k)_{k\in\mathcal{N}}, \mathcal{F}, (c_k)_{k\in\mathcal{N}}\right)$ *has a unique equilibrium for every* $(c_k)_{k\in\mathcal{N}} \in \prod_{k\in\mathcal{N}} [\underline{c}_k, \infty)$.

Proof. Let $L = \sum_{k\in\mathcal{N}} \lim_\infty h_k (> 0)$. For every $k \in \mathcal{N}$, let

$$\tilde{h}_k = h_k - \left(\lim_\infty h_k\right) + \frac{L}{|\mathcal{N}|}.$$

By Theorem 1, $\left(\mathcal{N}, \left(\tilde{h}_k\right)_{k\in\mathcal{N}}\right)$ is still a smooth and integrable demand component. In addition, it is easy to see that the demand system associated with smooth and integrable demand components $\left(\mathcal{N}, (h_k)_{k\in\mathcal{N}}\right)$ and $\left(\mathcal{N}, \left(\tilde{h}_k\right)_{k\in\mathcal{N}}\right)$ are identical. Therefore, for every $(c_k)_{k\in\mathcal{N}} \in \mathbb{R}^{\mathcal{N}}_{++}$, the sets of equilibria of pricing games $\left(\mathcal{N}, (h_k)_{k\in\mathcal{N}}, \mathcal{F}, (c_k)_{k\in\mathcal{N}}\right)$ and $\left(\mathcal{N}, \left(\tilde{h}_k\right)_{k\in\mathcal{N}}, \mathcal{F}, (c_k)_{k\in\mathcal{N}}\right)$ coincide. In the following, we work with $\left(\tilde{h}_k\right)_{k\in\mathcal{N}}$, and drop the tilde from the notations.

We want to show that condition (25) holds for every $f \in \mathcal{F}$ for some $(\underline{c}_k)_{k\in\mathcal{N}} \in \mathbb{R}^{\mathcal{N}}_{++}$. Let $f \in \mathcal{F}$. Let $(c'_k) \in \prod_{k\in f} \left(\underline{p}_k, \infty\right)$. Then, for every k in f, for every $x \geq c'_k$,

$$\theta_k(x) \leq \frac{1}{\chi_k(x)} \leq \frac{1}{\chi_k\left(c'_k\right)},$$

where the first inequality follows by Lemma 16. Put

$$\bar{\theta}^f = \max_{k\in f} \frac{1}{\chi_k\left(c'_k\right)}.$$

For every every $k \in f$, $\lim_\infty h_k > 0$, and therefore, $\lim_\infty \rho_k = \infty$. It follows that there exists $(\underline{c}_k)_{k\in f} \in \prod_{k\in f} (c'_k, \infty)$ such that for every $k \in f$, $\rho_k(x) \geq \bar{\theta}^f$ whenever $x \geq \underline{c}_k$. Therefore, for every $i, j \in f$, for every $x_i \geq \underline{c}_i$ and $x_j \geq \underline{c}_j$, $\rho_i(x_i) \geq \theta_j(x_j)$, and, in particular,

$$\frac{\omega^f \theta_i(x_i)}{1 - \omega^f \theta_i(x_i)} \left(1 - \omega^f \rho_j(x_j)\right) - \rho_i(x_i) < 0.$$

Therefore condition (25) holds for firm f and minimum cost profile $(\underline{c}_k)_{k\in f}$. Constructing such a minimum cost profile for every firm in $\mathcal{F}$ and applying Lemma 24 allows us to conclude. □

Proposition 19. *Let* $\left(\mathcal{N}, (h_k)_{k\in\mathcal{N}}, \mathcal{F}\right)$ *be a pricing structure satisfying Assumptions 1 and 2.*

For every $\underline{c} \in \mathbb{R}_{++}$*, there exists* $\underline{\alpha} \in \mathbb{R}_{+}$ *such that pricing game* $\left(\mathcal{N}, (h_k + \alpha_k)_{k\in\mathcal{N}}, \mathcal{F}, (c_k)_{k\in\mathcal{N}}\right)$ *has a unique equilibrium for every* $(\alpha_k)_{k\in\mathcal{N}} \in [\underline{\alpha}, \infty)^{\mathcal{N}}$ *and* $(c_k)_{k\in\mathcal{N}} \in [\underline{c}, \infty)^{\mathcal{N}}$*.*

Proof. Notice first that pricing structure $\left(\mathcal{N}, (h_k + \alpha_k)_{k\in\mathcal{N}}, \mathcal{F}\right)$ still satisfies Assumptions 1 and 2. Therefore, every pricing game based on this pricing structure has an equilibrium.

Let $\underline{c} > 0$. For every $c_k \geq \underline{c}$, for every $\mu^f \in (1, \bar{\mu}^f)$,[16]

$$r_k(\mu^f, c_k) > r_k(1, c_k) \geq r_k(1, \underline{c}) \equiv p_k^{\min} > \underline{p}_k.$$

Next, notice that θ_k does not depend on $(\alpha_k)_{k\in\mathcal{N}}$. For every $x \geq p_k^{\min}$,

$$\theta_k(x) \leq \frac{1}{\chi_k(x)} \leq \frac{1}{\chi_k\left(p_k^{\min}\right)} \equiv \bar{\theta}_k,$$

where the first inequality follows from Lemma 16. Put $\bar{\theta} \equiv \max_{k\in\mathcal{N}} \bar{\theta}_k$. On the other hand, ρ_k does depend on $(\alpha_k)_{k\in\mathcal{N}}$. We make this dependence explicit by writing ρ_k^α. For every $x \geq p_k^{\min}$,

$$\begin{aligned}\rho_k^\alpha(x) &= \frac{h_k(x) + \alpha_k}{\gamma_k(x)},\\ &\geq \frac{\alpha_k}{\gamma_k(x)},\\ &\geq \frac{\alpha_k}{\gamma_k\left(p_k^{\min}\right)}.\end{aligned}$$

Therefore, there exists $\underline{\alpha} \geq 0$ such that for every $(\alpha_k)_{k\in\mathcal{N}} \in [\underline{\alpha}, \infty)^{\mathcal{N}}$, for every $k \in \mathcal{N}$, for every $x \geq p_k^{\min}$, $\rho_k^\alpha(x) \geq \bar{\theta}$.

Next, fix a pricing structure $\left(\mathcal{N}, (h_k + \alpha_k)_{k\in\mathcal{N}}, \mathcal{F}\right)$, where $(\alpha_k)_{k\in\mathcal{N}} \in [\underline{\alpha}, \infty)^{\mathcal{N}}$. Let $f \in \mathcal{F}$, $\omega^f \in (0, \bar{\omega}^f)$ and $\mu^f = 1/(1-\omega^f)$. We want to show that condition (25) holds for firm f and lower bound $\underline{c}$. Let $i, j \in f$, $x_i \geq r_i\left(\mu^f, \underline{c}\right)$ and $x_j \geq r_j\left(\mu^f, \underline{c}\right)$. Notice that $x_i \geq p_i^{\min}$ and $x_j \geq p_j^{\min}$. Therefore, $\rho_j^\alpha(x_j) \geq \bar{\theta} \geq \theta_i(x_i)$, and

$$\begin{aligned}\omega^f \theta_i(x_i) \frac{1 - \omega^f \rho_j^\alpha(x_j)}{1 - \omega^f \theta_i(x_i)} - \rho_i^\alpha(x_i) &\leq \omega^f \theta_i(x_i) - \rho_i^\alpha(x_i),\\ &\leq \bar{\theta}(\omega^f - 1) < 0.\end{aligned}$$

Therefore, condition (25) holds for firm f and lower bound $\underline{c}$. By Lemma 24, pricing game $\left(\mathcal{N}, (h_k + \alpha_k)_{k\in\mathcal{N}}, \mathcal{F}, (c_k)_{k\in\mathcal{N}}\right)$ has a unique pricing equilibrium for every $(c_k)_{k\in\mathcal{N}} \in [\underline{c}, \infty)$.

[16] Notice that r_k does not depend on $(\alpha_k)_{k\in\mathcal{N}}$, since ι_k is not affected by $(\alpha_k)_{k\in\mathcal{N}}$.

□

Proposition 20. *Let $\left(\mathcal{N}, (h_k)_{k\in\mathcal{N}}, \mathcal{F}\right)$ be a pricing structure satisfying Assumptions 1 and 2. Assume that there exists a function h and scalars $(\alpha_k, \beta_k, \delta_k, \epsilon_k)_{k\in f} \in (\mathbb{R}_{++} \times \mathbb{R}_{++} \times \mathbb{R}_{+} \times \mathbb{R}_{+})^f$ such that for every $x > 0$, for every $k \in f$,*

$$h_k(x) = \alpha_k h\left(\beta_k x + \delta_k\right) + \epsilon_k.$$

Assume also that θ is non-increasing and ρ is non-decreasing on $(\underline{p}, \infty)$. Then, Assumption 4 holds for firm f.

Proof. Since ρ is non-decreasing and and θ is non-increasing, $\rho(x) \geq \theta(x)$ for every $x > \underline{p}$. Therefore, $\lim_{\underline{p}} \theta \leq \lim_{\underline{p}} \rho$, and for every $x > \underline{p}$,

$$\theta(x) \leq \lim_{\underline{p}} \theta \leq \lim_{\underline{p}} \rho \leq \rho(x).$$

Let i and j in l, $x_i > \underline{p}_i$ and $x_j > \underline{p}_j$. Let $k \in \{i, j\}$. Then,

$$\begin{aligned} \iota_k(x_k) &= \frac{\beta_k x_k}{\beta_k x_k + \delta_k} \iota\left(\beta_k x_k + \delta_k\right), \\ &\leq \iota\left(\beta_k x_k + \delta_k\right). \end{aligned}$$

Since $\iota_k(x_k) > 1$, it follows that $\iota\left(\beta_k x_k + \delta_k\right) > 1$. Therefore,

$$\theta_k(x_k) = \theta\left(\beta_k x_k + \delta_k\right) \leq \lim_{\underline{p}} \theta \leq \lim_{\underline{p}} \rho \leq \rho\left(\beta_k x_k + \delta_k\right) = \rho_k(x_k).$$

Therefore,

$$\theta_i(x_i) \leq \rho_j(x_j).$$

Therefore, Assumption 4 holds for firm f. □

Proposition 21. *Let $\left(\mathcal{N}, (h_k)_{k\in\mathcal{N}}, \mathcal{F}\right)$ be a pricing structure satisfying Assumptions 1 and 2. Assume that condition (11) holds for every firm in $\mathcal{F}$, and let $\alpha > 0$ and $(\beta_k, \delta_k, \epsilon_k)_{k\in\mathcal{N}} \in (\mathbb{R}_{++} \times \mathbb{R}_{+} \times \mathbb{R}_{+})^{\mathcal{N}}$. Let $\left(\widetilde{\mathcal{N}}, \left(\widetilde{h_k}\right)_{k\in\widetilde{\mathcal{N}}}, \widetilde{\mathcal{F}}\right)$, where $\widetilde{\mathcal{N}} = \mathcal{N}$, $\widetilde{\mathcal{F}} = \mathcal{F}$, and*

$$\widetilde{h_k}(x) = \alpha h_k\left(\beta_k x + \delta_k\right) + \epsilon_k,\ \forall x > 0,\ \forall k \in \mathcal{N}.$$

Then, $\left(\widetilde{\mathcal{N}}, \left(\widetilde{h_k}\right)_{k\in\widetilde{\mathcal{N}}}, \widetilde{\mathcal{F}}\right)$ is a pricing structure satisfying Assumptions 1 and 2, and condition (11) for every firm in $\widetilde{\mathcal{F}}$.

Proof. We already know from Proposition 6 that $\left(\widetilde{\mathcal{N}}, \left(\widetilde{h_k}\right)_{k\in\widetilde{\mathcal{N}}}, \widetilde{\mathcal{F}}\right)$ is a pricing structure satisfying Assumptions 1 and 2. Let $f \in \widetilde{\mathcal{F}}$, $\omega^f \in (0, \bar{\omega}^f)$ and $(\widetilde{x_k})_{k\in f}$ such that $\widetilde{\chi_k}\left(\widetilde{x_k}\right) > \omega^f$ for every $k \in f$. For every k in f, define $x_k = \beta_k \widetilde{x_k} + \delta_k$. As shown in the proof of Proposition 6, $\widetilde{\iota_k}\left(\widetilde{x_k}\right) \leq \iota_k(x_k)$. Therefore,

$$\chi_k(x_k) \geq \widetilde{\chi_k}\left(\widetilde{x_k}\right) > \omega^f.$$

Therefore, using again the formulas derived in the proof of Proposition 6,

$$\begin{aligned}
&\left(\sum_{k\in f} \frac{\omega^f \widetilde{\theta_k}\left(\widetilde{x_k}\right)}{1-\omega^f \widetilde{\theta_k}\left(\widetilde{x_k}\right)} \widetilde{\gamma_k}\left(\widetilde{x_k}\right)\right)\left(\frac{1}{\sum_{k\in f} \widetilde{h_k}\left(\widetilde{x_k}\right)} - \frac{\omega^f}{\sum_{k\in f} \widetilde{\gamma_k}\left(\widetilde{x_k}\right)}\right) \\
&= \left(\sum_{k\in f} \frac{\omega^f \theta_k\left(x_k\right)}{1-\omega^f \theta_k\left(x_k\right)} \alpha\gamma_k\left(x_k\right)\right)\left(\frac{1}{\sum_{k\in f} \alpha h_k\left(x_k\right)+\epsilon_k} - \frac{\omega^f}{\sum_{k\in f} \alpha\gamma_k\left(x_k\right)}\right), \\
&\leq \left(\sum_{k\in f} \frac{\omega^f \theta_k\left(x_k\right)}{1-\omega^f \theta_k\left(x_k\right)} \gamma_k\left(x_k\right)\right)\left(\frac{1}{\sum_{k\in f} h_k\left(x_k\right)} - \frac{\omega^f}{\sum_{k\in f} \gamma_k\left(x_k\right)}\right), \\
&< 0,
\end{aligned}$$

since $\chi_k(x_k) > \omega^f$ for every $k \in f$ and condition (11) holds for firm f in pricing structure $\left(\mathcal{N}, (h_k)_{k\in\mathcal{N}}, \mathcal{F}\right)$. □

C.11 Proof of Proposition 6.

Let $g \in \widetilde{\mathcal{F}}$. There exists $f(g) \in \mathcal{F}$ such that for all $j \in g$, there exists $k(j) \in f(g)$ and $(\alpha_j, \beta_j, \delta_j, \epsilon_j) \in \mathbb{R}^2_{++} \times \mathbb{R}_+$ such that

$$\forall x > 0,\ \widetilde{h}_j(x) = \alpha_j h_{k(j)}(\beta_j x + \delta_j) + \epsilon_j.$$

Then, for all $x > 0$,

$$\begin{aligned}
\widetilde{h}'_j(x) &= \alpha_j \beta_j h'_{k(j)}(\beta_j x + \delta_j) < 0, \\
\widetilde{h}''_j(x) &= \alpha_j \beta_j^2 h''_{k(j)}(\beta_j x + \delta_j) > 0, \\
\widetilde{\gamma}_j(x) &= \alpha_j \gamma_{k(j)}(\beta_j x + \delta_j), \\
\widetilde{\gamma}'_j(x) &= \alpha_j \beta_j \gamma'_{k(j)}(\beta_j x + \delta_j), \\
\widetilde{\rho}_j(x) &= \rho_{k(j)}(\beta_j x + \delta_j) + \frac{\epsilon_j}{\alpha_j \gamma_k(\beta_j x + \delta_j)} \geq \rho_{k(j)}(\beta_j x + \delta_j),
\end{aligned}$$

$$
\begin{aligned}
\widetilde{\theta}_j(x) &= \theta_{k(j)}(\beta_j x + \delta_j), \\
\widetilde{\iota}_j(x) &= \frac{\beta_j x}{\beta_j x + \delta_j} \iota_{k(j)}(\beta_j x + \delta_j).
\end{aligned}
$$

Therefore, $\left(\widetilde{\mathcal{N}}, \left(\widetilde{h}_k\right)_{k \in \widetilde{\mathcal{N}}}\right)$ is smooth and integrable, $\widetilde{\rho}_j \geq 1$ for every j in $\widetilde{\mathcal{N}}$, and Assumption 1 holds for $\left(\widetilde{\mathcal{N}}, \left(\widetilde{h}_k\right)_{k \in \widetilde{\mathcal{N}}}, \widetilde{\mathcal{F}}\right)$. Moreover, for every $j \in g$,

$$
\lim_{\infty} \widetilde{\iota}_j = \lim_{\infty} \iota_{k(j)} = \bar{\mu}^{f(g)}.
$$

Therefore, Assumption 2 also holds for $\left(\widetilde{\mathcal{N}}, \left(\widetilde{h}_k\right)_{k \in \widetilde{\mathcal{N}}}, \widetilde{\mathcal{F}}\right)$. Finally, we show that Assumption 4 also holds for firm $g \in \widetilde{\mathcal{F}}$. Let $j \in g$ and $x > \underline{\widetilde{p}}_j$. Then,

$$
1 < \widetilde{\iota}_j(x) \leq \iota_{k(j)}(\beta_j x + \delta_j).
$$

Therefore, $\beta_j x + \delta_j > \underline{p}_{k(j)}$, and

$$
\widetilde{\theta}_j(x) \leq \sup_{y > \underline{p}_{k(j)}} \theta_{k(j)}(y).
$$

It follows that $\sup_{y > \underline{\widetilde{p}}_j} \widetilde{\theta}_j(y) \leq \sup_{y > \underline{p}_{k(j)}} \theta_{k(j)}(y)$. Using the same reasoning, we also obtain that $\inf_{y > \underline{\widetilde{p}}_j} \widetilde{\rho}_j(y) \geq \inf_{y > \underline{p}_{k(j)}} \rho_{k(j)}(y)$. Therefore,

$$
\begin{aligned}
\max_{j \in g} \sup_{x > \underline{\widetilde{p}}_j} \widetilde{\theta}_j(x) &\leq \max_{j \in g} \sup_{x > \underline{p}_{k(j)}} \theta_{k(j)}(x), \\
&\leq \max_{k \in f(g)} \sup_{x > \underline{p}_k} \theta_k(x), \\
&\leq \min_{k \in f(g)} \inf_{x > \underline{p}_k} \rho_k(x), \\
&\leq \min_{j \in g} \inf_{x > \underline{p}_{k(j)}} \rho_{k(j)}(x), \\
&\leq \min_{j \in g} \inf_{x > \underline{\widetilde{p}}_j} \widetilde{\rho}_j(x).
\end{aligned}
$$

Therefore, Assumption 4 also holds for every g in $\widetilde{\mathcal{F}}$.

C.12 Establishing Equilibrium Uniqueness using Poincaré-Hopf

The reader may wonder whether we could obtain weaker uniqueness conditions by using more standard approaches. Uniqueness of a fixed point is usually established by using the contraction mapping approach, the univalence approach or the index (Poincaré-Hopf) approach. It is well known that the index approach is more general than the others, and that it provides an "almost if and only if" condition for uniqueness. We will therefore focus on the index approach. Since we will be working with matrices, we will sometimes assume that $\mathcal{F} = \{1, \dots, F\}$, and that firm f's set of products is $\mathcal{N}^f$.

We know that establishing uniqueness in the pricing game is equivalent to establishing uniqueness in the auxiliary game in which firms are simultaneously choosing their μ^f's. We also know that a profile $\mu = (\mu^f)_{f \in \mathcal{F}}$ is an equilibrium of the auxiliary game if and only if for every $f \in \mathcal{F}$,

$$\phi^f(\mu) \equiv (\mu^f - 1)\left(\left(\sum_{k \in \mathcal{N}^f} h_k\right) + \left(\sum_{\substack{g \in \mathcal{F} \\ g \neq f}} \sum_{k \in \mathcal{N}^f} h_k\right)\right) - \mu^f \sum_{k \in \mathcal{N}^f} \gamma_k = 0.$$

Therefore, all we need to do is show that map ϕ has a unique zero. By the index theorem, this holds if the determinant of the Jacobian matrix of ϕ evaluated at μ is strictly positive whenever $\phi(\mu) = 0$. We have shown in the proof of Lemma 10 that

$$\frac{\partial \phi^f}{\partial \mu^f} = \sum_{f \in \mathcal{F}} \sum_{k \in \mathcal{N}^f} h_k \equiv H(\mu).$$

Moreover, if $g \neq f$, then

$$\frac{\partial \phi^f}{\partial \mu^g} = (\mu^f - 1) \sum_{k \in \mathcal{N}^g} r'_k h'_k.$$

Therefore,

$$\begin{aligned}
\det J(\phi) &= \begin{vmatrix} H(\mu) & (\mu_1 - 1)\sum_{k \in \mathcal{N}^2} r'_k h'_k & \cdots & (\mu_1 - 1)\sum_{k \in \mathcal{N}^F} r'_k h'_k \\ (\mu_2 - 1)\sum_{k \in \mathcal{N}^1} r'_k h'_k & H(\mu) & \cdots & (\mu_2 - 1)\sum_{k \in \mathcal{N}^F} r'_k h'_k \\ \vdots & \vdots & \ddots & \vdots \\ (\mu^F - 1)\sum_{k \in \mathcal{N}^1} r'_k h'_k & (\mu^F - 1)\sum_{k \in \mathcal{N}^2} r'_k h'_k & \cdots & H(\mu) \end{vmatrix}, \\
&= \left(\prod_{f \in \mathcal{F}} (\mu^f - 1) \sum_{k \in \mathcal{N}^f} r'_k h'_k\right) \det \mathcal{M}\left(\left(1 + \frac{H(\mu)}{(\mu^f - 1)\sum_{k \in \mathcal{N}^f} r'_k(-h'_k)}\right)_{1 \leq f \leq F}\right),
\end{aligned}$$

where the second line has been obtained by dividing row f by $\mu^f - 1$ and dividing column f by $\sum_{k\in\mathcal{N}^f} r'_k h'_k$ for every f in $\{1,\dots,F\}$ and by using the F-linearity of the determinant. By Lemma 21,

$$\begin{aligned}
\det\left(J(\phi)\right) = & \left(\prod_{f\in\mathcal{F}}(\mu^f-1)\sum_{k\in\mathcal{N}^f} r'_k h'_k\right)(-1)^F \left(\left(\prod_{f\in\mathcal{F}}\left(1+\frac{H(\mu)}{(\mu^f-1)\sum_{k\in\mathcal{N}^f} r'_k(-h'_k)}\right)\right)\right.\\
& \left. -\sum_{g\in\mathcal{F}}\prod_{f\neq g}\left(1+\frac{H(\mu)}{(\mu^f-1)\sum_{k\in\mathcal{N}^f} r'_k(-h'_k)}\right)\right),\\
= & \left(\prod_{f\in\mathcal{F}}(\mu^f-1)\sum_{k\in\mathcal{N}^f} r'_k h'_k\right)(-1)^F \left(\prod_{f\in\mathcal{F}}\left(1+\frac{H(\mu)}{(\mu^f-1)\sum_{k\in\mathcal{N}^f} r'_k(-h'_k)}\right)\right)\\
& \times\left(1-\sum_{f\in\mathcal{F}}\frac{1}{1+\frac{H(\mu)}{(\mu^f-1)\sum_{k\in\mathcal{N}^f} r'_k(-h'_k)}}\right),\\
= & \underbrace{\left(\prod_{f\in\mathcal{F}}\left(H(\mu)+(\mu^f-1)\sum_{k\in\mathcal{N}^f} r'_k(-h'_k)\right)\right)}_{>0}\left(1-\sum_{f\in\mathcal{F}}\frac{1}{1+\frac{H(\mu)}{(\mu^f-1)\sum_{k\in\mathcal{N}^f} r'_k(-h'_k)}}\right).
\end{aligned}$$

Therefore, we need to show that

$$\sum_{f\in\mathcal{F}}\frac{\frac{\mu^f-1}{H(\mu)}\sum_{k\in\mathcal{N}^f} r'_k(-h'_k)}{1+\frac{\mu^f-1}{H(\mu)}\sum_{k\in\mathcal{N}^f} r'_k(-h'_k)} < 1 \tag{29}$$

whenever $\phi(\mu)=0$. Notice that

$$\begin{aligned}
(29) \iff & \sum_{f\in\mathcal{F}}\left(\frac{(\mu^f-1)\sum_{k\in\mathcal{N}^f} r'_k(-h'_k)}{1+\frac{\mu^f-1}{H(\mu)}\sum_{k\in\mathcal{N}^f} r'_k(-h'_k)}-\sum_{k\in\mathcal{N}^f} h_k\right)<0\\
\iff & \sum_{f\in\mathcal{F}}\left(\frac{(\mu^f-1)\sum_{k\in\mathcal{N}^f} r'_k(-h'_k)}{1+\frac{(\mu^f-1)^2}{\mu^f}\frac{\sum_{k\in\mathcal{N}^f} r'_k(-h'_k)}{\sum_{k\in\mathcal{N}^f}\gamma_k}}-\sum_{k\in\mathcal{N}^f} h_k\right)<0,\ \text{since } \phi(\mu)=0,\\
\iff & \sum_{f\in\mathcal{F}}\left(\frac{(\mu^f-1)\sum_{k\in\mathcal{N}^f} r'_k(-h'_k)}{1+\frac{\mu^f-1}{\mu^f}\frac{\sum_{k\in\mathcal{N}^f} r'_k\left((\mu^f-1)(-h'_k)-\mu^f(-\gamma'_k)+\mu^f(-\gamma'_k)\right)}{\sum_{k\in\mathcal{N}^f}\gamma_k}}-\sum_{k\in\mathcal{N}^f} h_k\right)<0,\\
\iff & \sum_{f\in\mathcal{F}}\left(\frac{(\mu^f-1)\sum_{k\in\mathcal{N}^f} r'_k(-h'_k)}{1-\frac{\mu^f-1}{\mu^f}+(\mu^f-1)\frac{\sum_{k\in\mathcal{N}^f} r'_k(-\gamma'_k)}{\sum_{k\in\mathcal{N}^f}\gamma_k}}-\sum_{k\in\mathcal{N}^f} h_k\right)<0,\ \text{by Lemma 7,}
\end{aligned}$$

$$\iff \sum_{f\in\mathcal{F}} \left(\frac{\mu^f(\mu^f-1)\sum_{k\in\mathcal{N}^f} r_k'(-h_k')}{1+\mu^f(\mu^f-1)\frac{\sum_{k\in\mathcal{N}^f} r_k'(-\gamma_k')}{\sum_{k\in\mathcal{N}^f}\gamma_k}} - \sum_{k\in\mathcal{N}^f} h_k \right) < 0,$$
$$\iff \Omega'\left(H(\mu)\right) < 0 \text{ (see the proof of Lemma 23).}$$

Therefore, the index approach gives us the exact same condition as the aggregative game approach.

D Proofs for Section 5

D.1 Proof of Proposition D.1

Using the definition of function S and equation (14), it is easy to see that $m(x)$ and $S(x)$ are jointly pinned down by:

$$m(x) = \frac{1}{\sigma-(\sigma-1)S(x)},$$
$$S(x) = x\left(1-m(x)\right)^{\sigma-1}.$$

Differentiating wrt x, we get:

$$\begin{aligned} m'(x) &= (\sigma-1)\left(m(x)\right)^2 S'(x), \\ S'(x) &= \left(1-m(x)\right)^{\sigma-1} - (\sigma-1)xm'(x)\left(1-m(x)\right)^{\sigma-2}. \end{aligned}$$

Solving out for S' and m' yields:

$$\begin{aligned} S'(x) &= \frac{\left(1-m(x)\right)^{\sigma-1}}{1+(\sigma-1)^2 x\left(1-m(x)\right)^{\sigma-2}\left(m(x)\right)^2} > 0, \\ m'(x) &= \frac{(\sigma-1)\left(m(x)\right)^2\left(1-m(x)\right)^{\sigma-1}}{1+(\sigma-1)^2 x\left(1-m(x)\right)^{\sigma-2}\left(m(x)\right)^2} > 0. \end{aligned}$$

Since $\pi = mS$, it follows that $\pi' > 0$.

Applying the implicit function theorem to equation $\Omega(H) = 1$ yields:

$$\frac{dH^*}{dT^f} = \frac{S'\left(\frac{T^f}{H^*}\right)}{\sum_{g\in\mathcal{F}}\frac{T^g}{H^*}S'\left(\frac{T^g}{H^*}\right)} > 0.$$

Next, notice that

$$\frac{d\left(\frac{T^f}{H^*}\right)}{d\theta^f} = \frac{1}{H^*}\left(1 - \frac{T^f}{H^*}\frac{dH^*}{dT^f}\right),$$
$$= \frac{1}{H^*}\left(1 - \frac{\frac{T^f}{H^*}S'\left(\frac{T^f}{H^*}\right)}{\sum_{g\in\mathcal{F}}\frac{T^g}{H^*}S'\left(\frac{T^g}{H^*}\right)}\right) > 0,$$

and that, for $g \neq f$,

$$\frac{d\left(\frac{T^g}{H^*}\right)}{d\theta^f} = -\frac{T^g}{H^{*2}}\frac{dH^*}{dT^f} < 0.$$

Therefore, points (ii) and (iii) follow immediately by applying the chain rule.

D.2 Proof of Proposition 10

Proof. Applying the implicit function theorem to equation (16), we see that, for every $x > 0$,

$$m'(x) = \frac{m(x)e^{-m(x)}}{\frac{1}{m(x)} + m(x)\,xe^{-m(x)}} > 0.$$

Notice also that $S(x) = x\,\mathrm{e}^{-m(x)} = 1 - \frac{1}{m(x)}$. Therefore, $S' > 0$ and $\pi' > 0$.

Points (ii) and (iii) follow by applying the implicit function theorem to equation

$$\sum_{f\in\mathcal{F}} S\left(\frac{T^f}{H}\right) = 1$$

as we did in D.1. □

E Proofs for Section 6

E.1 Proof of Proposition 11

Proof. Suppose first that there are no synergies, i.e., $\eta = 0$. We claim that the fitting-in functions satisfy $\hat{m}^{f\cup g}(H) > \max\left(m^f(H), m^g(H)\right)$ for all $H > 0$. We now show that $\hat{m}^{f\cup g}(H) > m^f(H)$. (The argument as to why $\hat{m}^{f\cup g}(H) > m^g(H)$ is the same.) We have

$$1 \quad = \quad \hat{m}^{f\cup g}\left(1 - \frac{\sum_{j\in f\cup g}\gamma_j\left(r_j(\hat{m}^{f\cup g}, c_j)\right)}{H}\right)$$

$$
\begin{aligned}
&= m^f \left(1 - \frac{\sum_{j\in f} \gamma_j \left(r_j(m^f, c_j)\right)}{H}\right) \\
&> m^f \left(1 - \frac{\sum_{j\in f\cup g} \gamma_j \left(r_j(m^f, c_j)\right)}{H}\right)
\end{aligned}
$$

where the equalities follow from Lemma 14. As the right-hand side is increasing in the ι-markup μ, it follows that $\hat{m}^{f\cup g}(H) > m^f(H)$. Hence, for any H, the contribution of the merged firm to the aggregator is smaller than the sum of the contributions of the merger partners pre-merger:

$$
\sum_{j\in f\cup g} h_j(r_j(\hat{m}^{f\cup g}(H))) < \sum_{j\in f} h_j(r_j(m^f(H))) + \sum_{j\in g} h_j(r_j(m^g(H)))
$$

As the fitting-in function of any firm not participating in the merger (and thus its contribution to the aggregator) is unaffected, this implies that $\hat{\Omega}(H) < \Omega(H)$ for all $H > 0$. Recall from the proof of Lemma 15 that $\Omega(H) > 1$ for H sufficiently small and $\Omega(H) < 1$ for H sufficiently large. Hence, in both the smallest and largest equilibrium, $\hat{H}^* < H^*$. That is, in the absence of synergies ($\eta = 0$), the merger is CS-decreasing.

As a decrease in marginal cost induces an increase in consumers' indirect utility by Proposition 3, it follows that there exists a cutoff $\hat{\eta} > 0$ such that the merger is CS-increasing if and only if $\eta > \hat{\eta}$. (If $\hat{\eta} = 1$, then even maximal synergies do not suffice to make the merger CS-increasing.)

We also claim that, in the absence of synergies ($\eta = 0$), the merger is privately profitable and induces an increase in all prices. To see this, note that as $\hat{H}^* < H^*$, $r_k'(\cdot) > 0$, and $m'(\cdot) < 0$, the prices of all products of outsiders to the merger are higher after the merger than before, implying that $\hat{H}^{*-(f\cup g)} < H^{*-(f\cup g)}$. So, even if the merged firm were to charge the same prices as the merger partners did before the merger, the merger would be privately profitable. But, in fact, the merged firm choses to charge higher prices than its partners did before the merger. This is for two reasons. First, because in the absence of synergies, $\hat{m}^{f\cup g}(H) > \max\left(m^f(H), m^g(H)\right)$ for all $H > 0$. Second, because in the absence of synergies, the merger induces an increase in the equilibrium level of the aggregator. □

E.2 Proof of Lemma 20

Proof. Using the notation introduced in Section 5, let

$$\hat{T}^M \equiv H^* S^{-1}\left(S\left(\frac{T^f}{H^*}\right)+S\left(\frac{T^g}{H^*}\right)\right). \tag{30}$$

If $T^M = \hat{T}^M$, we have:

$$\begin{aligned} 1 &= \sum_{l\in\mathcal{F}} S\left(\frac{T^l}{H^*}\right) \\ &= S\left(\frac{T^M}{H^*}\right) + \sum_{l\in\mathcal{F}\setminus(f\cup g)} S\left(\frac{T^l}{H^*}\right), \end{aligned}$$

where the first equality is the pre-merger equilibrium condition whereas the second equality follows from $T^M = \hat{T}^M$. As equilibrium is unique when demand is CES or multinomial logit, we have $\hat{H}^* = H^*$. That is, the merger is CS-neutral if $T^M = \hat{T}^M$. As $S'(\cdot) > 0$, if $T^M > \hat{T}^M$, we have

$$S\left(\frac{T^M}{H^*}\right) + \sum_{l\in\mathcal{F}\setminus(f\cup g)} S\left(\frac{T^l}{H^*}\right) > 1,$$

implying that $\hat{H}^* > H^*$, so the merger is CS-increasing. Similarly, if $T^M < \hat{T}^M$, then $\hat{H}^* < H^*$, so the merger is CS-decreasing.

Next, we note that a CS-neutral merger involves synergies in that $\hat{T}^M > T^f + T^g$. Suppose otherwise that $\hat{T}^M \leq T^f + T^g$. Then,

$$\begin{aligned} S\left(\frac{\hat{T}^M}{H^*}\right) &\leq S\left(\frac{T^f + T^g}{H^*}\right) \\ &< S\left(\frac{T}{H^*}\right) + S\left(\frac{T^g}{H^*}\right), \end{aligned}$$

where the first inequality follows from $S'(\cdot) > 0$ and the second from $S''(\cdot) < 0$, as can be verified to hold under both CES and multinomial logit demands. But then the merger would be CS-decreasing, a contradiction. Hence, $\hat{T}^M > T^f + T^g$.

To see that a CS-neutral merger is profitable, note that:

$$\pi\left(\frac{\hat{T}^M}{H^*}\right) > m\left(\frac{\hat{T}^M}{H^*}\right) S\left(\frac{\hat{T}^M}{H^*}\right)$$

$$
\begin{aligned}
&= m\left(\frac{\hat{T}^M}{H^*}\right)\left[S\left(\frac{T^f}{H^*}\right)+S\left(\frac{T^g}{H^*}\right)\right]\\
&> m\left(\frac{T^f}{H^*}\right)S\left(\frac{T^f}{H^*}\right)+m\left(\frac{T^g}{H^*}\right)S\left(\frac{T^g}{H^*}\right)\\
&= \pi\left(\frac{T}{H^*}\right)+\pi\left(\frac{T^g}{H^*}\right),
\end{aligned}
$$

where the second equality follows because the merger is CS-neutral, and the second inequality follows because $\hat{T}^M > T^f + T^g > \max\left(T^f, T^g\right)$ and $m'(\cdot) > 0$, both under CES and multinomial logit demands. Hence, merger M is profitable if $T^M = \hat{T}^M$.

Next, consider the effect of an increase in firm type T^M on the equilibrium level of the aggregator H^*. Applying the implicit function theorem to $\Omega(H^*) \equiv \sum_{u\in\mathcal{F}} S\left(\frac{T^u}{H^*}\right) = 1$, we obtain:

$$
\frac{dH^*}{dT^M} = \frac{S'\left(\frac{T^M}{H^*}\right)}{\sum_{u\in\mathcal{F}}\frac{T^u}{H^*}S'\left(\frac{T^u}{H^*}\right)} > 0,
$$

where the inequality follows as $S'(\cdot) > 0$. The effect of an increase in T^M on M's equilibrium profit is thus given by

$$
\begin{aligned}
\frac{d\pi\left(\frac{\hat{T}^M}{H^*}\right)}{T^M} &= \frac{\pi'\left(\frac{T^M}{H^*}\right)}{H^*}\left(1-\frac{T^M}{H^*}\frac{dH^*}{dT^M}\right)\\
&= \frac{\pi'\left(\frac{T^M}{H^*}\right)}{H^*}\left(1-\frac{T^M}{H^*}\frac{S'\left(\frac{T^M}{H^*}\right)}{\sum_{u\in\mathcal{F}}\frac{T^u}{H^*}S'\left(\frac{T^u}{H^*}\right)}\right) > 0,
\end{aligned}
$$

where the inequality follows as $\pi(\cdot) = m(\cdot)S(\cdot)$, $m'(\cdot) > 0$ and $S'(\cdot) > 0$, implying that $\pi'(\cdot) > 0$. That is, an increase in a firm's type induces an increase in that firm's equilibrium profit. It follows that merger M is profitable if $T^M \geq \hat{T}^M$, i.e., it is CS-nondecreasing. □

E.3 Proof of Proposition 12

We first prove the following lemma:

Lemma 26. *In a multiproduct-firm pricing game with CES or Logit demands, $\varepsilon' < 0$, where $\varepsilon(x) = x\frac{S'(x)}{S(x)}$ for all $x > 0$.*

Proof. Under CES demands,

$$
\varepsilon(x) = \frac{1}{1+(\sigma-1)^2 x(1-m(x))^{\sigma-2}m(x)^2},
$$

$$= \frac{1}{1 + (\sigma - 1)^2 S(x) \frac{m(x)^2}{1 - m(x)}},$$

which is indeed decreasing in x, since m and S are decreasing.

Under Logit demands,

$$\varepsilon(x) = \frac{1}{1 + m(x)^2 S(x)},$$

which is also decreasing in x, since m and S are decreasing. □

We can now prove Proposition 12:

Proof. We first show that $d\hat{T}^M/dH^* < 0$. Differentiating equation (30), we obtain

$$\begin{aligned}
S'\left(\frac{\hat{T}^M}{H^*}\right)\frac{d\hat{T}^M}{dH^*} &= \frac{\hat{T}^M}{H^*}S'\left(\frac{\hat{T}^M}{H^*}\right) - \frac{T^f}{H^*}S'\left(\frac{T^f}{H^*}\right) - \frac{T^g}{H^*}S'\left(\frac{T^g}{H^*}\right), \\
&= \varepsilon\left(\frac{\hat{T}^M}{H^*}\right)S\left(\frac{\hat{T}^M}{H^*}\right) - \varepsilon\left(\frac{T^f}{H^*}\right)S\left(\frac{T^f}{H^*}\right) - \varepsilon\left(\frac{T^g}{H^*}\right)S\left(\frac{T^g}{H^*}\right), \\
&= \varepsilon\left(\frac{\hat{T}^M}{H^*}\right)\left(S\left(\frac{T^f}{H^*}\right) + S\left(\frac{T^g}{H^*}\right)\right) - \varepsilon\left(\frac{T^f}{H^*}\right)S\left(\frac{T^f}{H^*}\right) - \varepsilon\left(\frac{T^g}{H^*}\right)S\left(\frac{T^g}{H^*}\right), \\
&< 0,
\end{aligned}$$

where the third line follows by definition of $\hat{T}^M$ and the last line follows from Lemma 26 and from the fact that $\hat{T}^M > T^f + T^g$.

Suppose M_i is CS-nondecreasing in isolation, which means that $T^{M_1} \geq \hat{T}^{M_1}$. If the CS-nondecreasing merger M_j takes place, the equilibrium value of the aggregator H^* weakly increases, and so the cutoff $\hat{T}^{M_1}$ weakly decreases. As T^{M_1} was initially above the cutoff, it therefore remains so after M_j has taken place, i.e., M_i is still CS-nondecreasing. A similar argument can be used to show the sign-preserving complementarity for mergers that are CS-decreasing in isolation. □

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