Collusion with a Greedy Centre in Sponsored Search Auctions

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Abstract

In this paper we aim at studying the sensitivity of the Generalized Second-Price auction to bidder collusion when monetary transfers are allowed. We propose a model of position auction that incorporates third parties as agents facilitating collusion in complete information. We show that the first-best collusive outcome can be achieved under any Nash condition. Under the *locally envy-free criterion*, we find that if the collusive gain is uniformly redistributed among members, the best that can be achieved is *Vickrey-Clarkes-Groves* outcome. Bidders do not have sufficient incentives to reduce even more their expressed demand. We then provide elements upon which an incentive compatible fee can be set by the centre in order to give the highest utility to advertisers subject to maximizing his own profits. We provide conditions under which bidders can enhance efficient collusion. Doing so we also contribute to the literature on collusion in multiple-objects simultaneous auctions.

Keywords: Auctions; Online advertising; Position auctions; Bidding ring; Cartel

JEL: D44, C72, M3, L41

1 Introduction

Collusion is one of the most long-standing issue for auction design as being the most prominent threat to the seller's revenue. This notion refers essentially to any attempt to engage in anti-competitive behaviour to suppress rivalry and to get an extra rent that would have otherwise been captured by the seller. As a result, it may alter both the allocative efficiency of the mechanism and the revenue from the auction. We are interested in the bidders' collusion that actually occurs in one of the auctions with the broadest scope ever, namely the *Generalized Second-Price* auction (*GSP*) run daily on the internet by search engines. Each time a customer enters a search query, an automated auction is triggered putting advertisers in competition for ad spaces. As this market generates billions of dollars the question whether the *GSP* auction proves to be robust (or not) to bidding rings is of great practical concern.

In online advertising, an increasing number of advertisers delegate their search marketing and paid search strategies to third parties specialized in such activities[1]. As a result, these third parties,

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[1]See the Gartner's U.S. Digital Marketing Spending report (2013). On the 243 firms envolved in the study, more than 50% of the respondents delegated their search marketing activities.

to which we refer as *intermediary firms,* will act on behalf of different advertisers during the same keyword auctions. We can see them as devices organizing collusion among different advertisers. This could be a problematic issue as these firms are free to affect the advertisers' final payments negatively by using coordination. This paper aims to analyse the impact of such a delegation on the performance of the *GSP* mechanism and under which conditions bidders can implement an efficient collusive mechanism.

We introduce a model of collusion based on side payments in the one-shot *GSP* auction with two positions and three players. The *intermediary firm* is assumed to be risk-neutral, incentiveless (in the sense that he maximizes the surplus of the members of the cartel) and to support no unit cost. However, this firm will seek to exploit the spoils of collusion. He will settle individual contracts in which he makes a proposal of a bid profile to be played and announces that an *ad hoc* fee is fixed towards internal monetary transfers (among members). It is assumed that (*i*) only one *intermediary firm* operates on the market and that no collusive agreement can be settled without a contract through this agent[2] and that (*ii*) the cartel is all-inclusive, which avoids the issue of competitive bids and cooperative bids interactions[3].

Under this specification, we find that (*i*) the *intermediary firm* can coordinate bidders on an *efficient* equilibrium which entails surplus that are beyond the convex hull of Nash equilibrium payoffs by pinning down the outcome to a *low-revenue* equilibrium and (*ii*) it can extract an optimal fee towards the spoils of the collusion without breaking incentives to collude (propositions (1) and (2)). In addition, we shown that, if the centre becomes absolutely greedy, the unique collusive outcome becomes the *lowest* Nash equilibrium, which implies both an upper boundary on bidders' payoffs utility functions (corollary (1)) and on the auctioneer's revenues (proposition (3)). The *non-cooperative* outcome can thus be sustained as a collusive one.

From the seminal studies by Aggarwal et al. (2006), Lahaie (2006), Varian (2007), Edelman et al. (2007), Athey and Ellison (2011) it is well known that the GSP auction involves a plethora of Nash equilibria (e.g., Börgers et al. (2013)). The strength of Varian (2007), Edelman et al. (2007) was to show that, by introducing a tractable refinement called *locally envy-free* (*LEF*) or *symmetric* equilibrium, the mechanism ends up with a unique outcome strictly equivalent in price and allocation to what would have implement the well known VCG mechanism. Unfortunately, the strategic choice for this refinement is not fairly justified and neither does the VCG-equivalent outcome. The main focus of the literature has by now relied on the theoretical justification of this result[4].

We take the counterpart of the *non-cooperative* literature and learn that, delegation of bids

[2] As the auction game takes place on a virtual market, it is assumed that advertisers does not have the ability to communicate directly.

[3] It appears for us to be a sufficient condition to analyse efficient and profitable collusion. Such an environment is optimal as the intermediary firm will only have to control for its members' bids and not anticipate bids of potential outside bidders. Then, if a profitable and efficient collusion cannot be implemented under the all-inclusive assumption, one can reasonably assume that it will not be the case if we allow for incomplete cartels.

[4] See for instance Cary et al. (2007), Edelman and Schwartz (2010), Milgrom (2010), Athey and Ellison (2011), Lucier et al. (2012), Börgers et al. (2013), Hashimoto (2013)

coupled with a rent seeker *intermediary firm* (i) rules out the multiplicity of equilibria by selecting the lowest Nash equilibrium in price, which implies an upper boundary on bidders' surplus, (ii) selects the *VCG-equivalent* outcome as the unique one if side contracts are based on the *LEF* criterion. This offers a natural justification motivated by explicit coordination. The last observation sustains the robustness of the stability criterion of Varian (2007), Edelman et al. (2007) and sustains the robustness of the mechanism to collusion. Bidders do not have sufficient incentives to reduce their expressed demand and the best that can be achieved is the *non-cooperative* outcome.

To the best of our knowledge only two papers offer an explicit representation of coordinated bidding in two different frameworks. The first paper, by Ashlagi et al. (2009), explicitly implement a coordinating device to the position auction through the solution concept of *mediated equilibria*. Authors consider Bayes-Nash equilibria and introduce a third-party, called a *mediator*, which, by using the revelation principle, implements the *VCG* outcome as the uniquely possible one. They extend this result to a broader class of position auctions, including the *generalized first-price* auction and the *generalized* m^{th}*-price* auction. This mediator acts in the same way as the third party we introduce with the restriction of no side contracts. We depart from this analysis as we allow transfers among bidders despite the similar result we obtain. Our analysis stems from the application of the LEF criterion along with position-dependent side payments. Moreover, we find it more realistic to assume that the intermediary firm gives a monetary payback to each advertiser with whom it has a bilateral contract. The second paper, by Vorobeychik and Reeves (2008), considers collusion in the context of repeated position auctions in a different way from the above authors. Based on the work by Cary et al. (2008) and the myopic best-response they consider, Vorobeychik and Reeves (2008) show that there exists a collusive equilibrium supported by strict *non-cooperative* behaviours. In their equilibrium, each player bids an infinitesimal quantity above the valuation of the first player not assigned to the list of positions and this equilibrium strategy profile is supported by a specific punishment threat. In case of a defection from a bidder, the remaining one automatically triggers the *VCG* equilibrium profile which implies a strict loss for the defector. In contrast to this work, our model suggests that, except for the limited case cited above, in any collusive equilibrium it is optimal for each member to bid a quantity strictly below the valuation of this lowest-valuing player. It has to be noted that, contrary to Vorobeychik and Reeves (2008), the collusion takes place in a one-shot *GSP* auction with monetary transfers and without any grim-trigger strategy.

The *GSP* mechanism is related to the simultaneous ascending-price auction, which has been found to be highly conductive to collusive behaviours[5]. Does this selling mechanism import this weakness? We find that the *low-revenue* property of the simultaneous ascending-price auction, established by Engelbrecht-Wiggans and Kahn (2005) and Milgrom (2000)[6], also emerges at the *first-best collusive outcome* of the *GSP* auction. However, it vanishes whenever the positions are

[5]See Cramton and Schwartz (2000, 2002), Brusco and Lopomo (2002), Milgrom (2000) Kwasnica (1998), Milgrom (2000), Ausubel et al. (2014).

[6]Milgrom (2000) has shown under the assumption of common knowledge valuations that a simple sequential equilibrium resulting in the lowest prices allowed by the auction exists. The outcome reaches the smallest equilibrium possible but is inefficient as units are split between buyers and are not won by the highest-valuing one. See theorem 7-8.

seen as perfect substitutes and/or whenever the centre becomes too greedy when extracting a rent from the collusion (propositions (1) and corollary (1))[7]. This *low price* property is also linked to the *demand reduction* pathology of standard multi-units auctions and relies on the multi-units demand assumption[8]. If bidders ask for one unit at the most, as for the GSP auction, this pathology should disappear. Despite this, *demand reduction* is a reluctant phenomena for the GSP auction, it is not optimal to bid one's own true valuation for a click. Nevertheless, under certain conditions on the click ratio this pathology can be avoided.

More precisely, there exists a tight link between the click ratio between positions and the competitive nature between players. If the gap in clicks is such that the top position is the most attractive, then the game reduces strategically to a standard Vickrey auction where it becomes optimal to be one's own valuation. In contrast, if the gap in clicks is such that positions become perfect substitutes, then the shading drops to an extreme level as the game results in a standard Bertrand competition. In the present work, however, if positions are quite different, then incentives converge to those observed in collusion in standard second-price auction when side contracts are allowed. The competition is essentially about the top position and in contrast it cannot reach a lower level than the one in the *lowest* Nash equilibrium. Hence, under our specific context the phenomenon of *demand reduction* remains pathological and permanent even if bidders are single unit-demanders.

Finally, the main literature on bidding rings focuses essentially on the *intermediary firm*'s ability to extract private information from members of the cartel[9]. By retaining Varian (2007), Edelman et al. (2007)'s complete information setting, we mechanically drive out this issue as the *intermediary firm* has full knowledge about each member's private valuations.

The paper is organized as follows: section (2) describes the model, in which we introduce collusive device involving side-payments by following Graham and Marshall (1987) and we explain the main issue the *intermediary firm* faces. Section (3) presents the *non-cooperative* benchmark. Section (4) presents the main results and offers a justification of the *VCG*-equivalent outcome as a unique ending point.

2 The model

2.1 The position auction game with collusion

We model the position auction as a game involving an all-inclusive cartel (or *ring*) of 3 bidders from a set $\mathcal{N} = \{1, 2, 3\}$ who compete for 2 positions from a set $\mathcal{K} = \{1, 2\}$ simultaneously sold by a search engine. A third-party who plays the role of what we call the *intermediary firm* or centre

[7] This result also asserts the idea that in complete information the standard results by Graham and Marshall (1987), McAfee and McMillan (1992) can be exported to this particular auction mechanism.

[8] Buyers have incentives to shade their bids with respect to their marginal valuation for each subsequent unit. Their first bid affect the price they pay on each unit at the end.

[9] See for instance Graham and Marshall (1987), Mailath and Zemsky (1991), McAfee and McMillan (1992), Marshall and Marx (2007)

coordinates the collusion. Each position k returns a commonly known expected *click-through rate* (henceforth *ctr*) denoted by $\alpha_k \geq 0$ with $\alpha_1 > \alpha_2 > \alpha_3 = 0$. A player's valuation expresses his willingness to pay for a click and is denoted by x_i, with $x_i \in [0, \bar{x}]$. Let $\mathbf{x} = \{x_1, x_2, x_3\}$ be the set of individual valuations labelled in decreasing order so that $x_1 > x_2 > x_3$. We assume that valuations are independent of positions, identities of advertisers allocated and of customers' clicking behaviours[10]. Henceforth, i's value of being allocated in position 1 or 2 is defined by the product $\alpha_k x_i$.

In the auction game, each player will simultaneously choose a uni-dimensional and non-negative bid b_i for a click as a function of his own valuation. To distinguish between a *non-cooperative* (or *Nash*) bid and a *collusive* one we denote by b_i and by $b_i^{\mathcal{N}}$ the former and the latter respectively. Then, let $\boldsymbol{B} = \{b_1, b_2, b_3\}$ be the set of bids and $\mathbf{B}_{-i}$ the set of bids except b_i and let $\boldsymbol{B}^{\mathcal{N}} = \left\{b_1^{\mathcal{N}}, b_2^{\mathcal{N}}, b_3^{\mathcal{N}}\right\}$ be the set of possible actions for ring members, with $\boldsymbol{B}_{-i}^{\mathcal{N}}$ defined in a similar fashion.

The game consists of allocating positions to bidders based on the order of their bids (or *expressed demand*). Let $\iota(k)$ be the identity of the bidder in position k, then, according to the rule of the *GSP,* he will be charged a price *per* click $p_k = \alpha_k b_{\iota(k+1)}$. Thus, the *non-cooperative* payoff utility function of a player $\iota(k)$ if assigned a position k equals to

$$\pi_{\iota(k)} = \alpha_k \left(x_{\iota(k)} - b_{\iota(k+1)}\right)$$

Consider that bidders meet before the main auction starts. If they all attempt to outbid each other during the main auction they will give most of the surplus to the seller. The task is then to find an agreement to limit the surplus extraction and to expropriate a significant share of it from the seller. The objective of the *intermediary firm* is to manage this coordination so that no bidder can find it profitable to shatter the settlement by reversing to competitive behaviour. The goal is thus to elaborate individual contracts $\boldsymbol{\gamma} = \{\gamma_1, \gamma_2, \gamma_3\}$ so that each potential member is better off than in the absence of coordination.

We assume that the sustainability of the cartel (or *ring*) is based on the presence of side-payments. That is, there is a monetary transfer $\omega_{\iota(k)}$, corresponding to a bidder's contribution, from each member to the *intermediary firm.* In return the centre makes a lump-sum transfer τ to each member of the collusive benefits[11].

Remark 1. Individual contributions are based on the anticipated final allocation and supported as a *sunk cost* by members. Each has to pay his own contribution no matter what the final allocation is.

These payments are computed in a standard fashion as being the difference between the price

[10]This is a highly simplified assumption. We do not consider the question of the allocative externality generated, for instance, by a firm's reputation among customers. A high-reputed firm might imply more clicks for low-reputed one if the former is placed on a position just above. It can be observed that some types of customers, more experienced, click more carefully and also more frequently on ads placed in the median position. The quality of the ad and the reputation of firms affect their choices. Then one could allow valuations for clicks to vary non-linearly between positions.

[11]To be clear, the latter is done at the end of the main auction so that the third party holds the entire bargaining power and affects necessarily the incentives compatibility constraints.

a bidder i would have to pay to the seller in the absence of a *cartel* and the price he actually pays with the *cartel* operating. Let $\hat{p}_k$ be the payment of member i to the seller when assigned a position k when the *ring* operates. Then, $\omega_{\iota(k)}$ takes the following simple formulation:

$$\omega_{\iota(k)} \quad = \quad \max\{p_k - \hat{p}_k; 0\} = \max\left\{\alpha_k\left(b_{\iota(k+1)} - b^{\mathcal{N}}_{\iota(k+1)}\right); 0\right\} \tag{1}$$

The total expected gain $\Pi_{\mathcal{N}}$ of the collusion (referred to hereafter as *spoil*), to be *uniformly* redistributed at the end of the auction, is then the sum of each member's payment to the *intermediary firm*:

$$\Pi_{\mathcal{N}} = \sum_{k=1}^{2} \omega_{\iota(k)} = \sum_{k=1}^{2} (p_k - \hat{p}_k) \tag{2}$$

At the end of the auction, the centre can credibly exert his bargaining power by setting a fee $\varepsilon \in [0,1]$ towards $\Pi_{\mathcal{N}}$. He can behave in three ways: (i) sporting the role of a risk-neutral incentiveless agent when $\varepsilon = 0$ or (ii) levying a fee $0 < \varepsilon < 1$ towards the collusive profits, or (iii) keeping the entire collusive gain with $\varepsilon = 1$ and substituting the seller. The quantity ε is set *non-strategically* and each member receives a transfer τ equals to:

$$\tau = \frac{(1-\varepsilon)}{3} \sum_{k=1}^{2} \omega_{\iota(k)} = \frac{1-\varepsilon}{3} \sum_{k=1}^{2} \alpha_k \left(b_{\iota(k+1)} - b^{\mathcal{N}}_{\iota(k+1)}\right) \tag{3}$$

We call the individual quantity in (3) a *uniform-ε redistribution rule* and it is assumed to be retained by the centre if a defection occurs. Note that τ is independent as i's own allocation. Given the redistribution scheme, the total payment of player i to the cartel will be function $p^{\mathcal{N}}_i : \mathbb{R}^{\mathcal{N}} \mapsto \mathbb{R}^+$ so that given allocation ι, individual payments take the following formulation:

$$p^{\mathcal{N}}_k = \begin{cases} \omega_{\iota(k)} - \tau & \text{if } k = \{1,2\} \\ -\tau & \text{if } k = \{\emptyset\} \end{cases} \tag{4}$$

that is, bidder $\iota(k)$ if assigned to position $k = \{1,2\}$ pays to the cartel his individual contribution and receives his individual share from the centre as returns, whereas if $k = \{\emptyset\}$, he only gets his individual share as he does not contribute[12]. As a result, a member's expected gain equals:

$$\pi^{\mathcal{N}}_{\iota(k)} = \begin{cases} \alpha_k \left(x_{\iota(k)} - b^{\mathcal{N}}_{\iota(k+1)}\right) - p^{\mathcal{N}}_k & \text{if } k = \{1,2\} \\ -\tau & \text{if } k = \{\emptyset\} \end{cases}$$

[12] The environment of complete information highly simplifies the functioning of such a redistribution rule as there is no need to implement a *pre-knockout* in order to make players reveal their private willingness to pay for a click. There is no adverse selection in this framework and no issue of cartel misrepresentation at the main auction.

The surplus of the *intermediary firm* is simply equal to the following:

$$\Gamma_{\mathcal{N}} = \sum_{k=1}^{2} \omega_{\iota(k)} - \sum_{i=1}^{3} \frac{(1-\varepsilon)}{3} \sum_{k=1}^{2} \omega_{\iota(k)} = \varepsilon \sum_{k=1}^{2} \omega_{\iota(k)}$$

The *intermediary firm* offers to each bidder a contract, which he is committed to, composed of: a system of recommended bids $\boldsymbol{\mu} = \left(b_i^{\mathcal{N}}\right)_{i\in\mathcal{N}}$ to be played during the auction, the share $\varepsilon \in [0,1]$ it intends to keep and the side-payments required from each member $\mathbf{p}^{\mathcal{N}} = \left(p_i^{\mathcal{N}}\right)_{i\in\mathcal{N}}$. For the sake of simplicity, we assume that the seller is passive by setting a reserve price equal to zero and that each bidder cannot act as an intermediary firm by reselling his position afterwards.

2.2 Coordination issue faced by the centre

Before the auction starts the centre makes the contract proposal $\boldsymbol{\gamma}_i = \left(b_i^{\mathcal{N}}, p_i^{\mathcal{N}}, \varepsilon\right)$. Each member accepts the proposal or rejects it. This is referred to as the *participation phase.* Members are then asked to pay their contributions ω_i and to bid at the main auction. They can act according to the recommendation, i.e., bid $\mu_i = b_i^{\mathcal{N}}$, or they can defect the agreement and bid $\tilde{b}_i$, double-crossing the *ring.* This will be the *deviation phase.* The seller then allocates the bidders in decreasing order of their bids and each is charged $\hat{p}_i(\mathbf{x})$. If no defection occurred during the auction stage, the centre redistributes τ and nothing else.

Assumption 1. *If one rejects the proposal then, the ring does not form, otherwise, the ring operates.*

We use this strong assumption in order to avoid the tractability issue inherent to the coexistence of both cooperative bids and non-cooperative one. The *intermediary firm* will only have to control for its members' bids and not anticipate bids of potential outside bidders

The center's problem of maximizing the collusive surplus can be written as follows: maximize the total (*ring*) surplus to the n members:

$$\max_{b_1^{\mathcal{N}}, b_2^{\mathcal{N}}, b_3^{\mathcal{N}}} BS \quad = \quad \sum_{i=1}^{3} \pi_i^{\mathcal{N}} \left(b_i^{\mathcal{N}}, \mathbf{B}^{\mathcal{N}}, x_i\right) \tag{5}$$

subject to incentive compatibility, $\forall i = \{1,2,3\}$, $\forall b_i^{\mathcal{N}} \in \mathbf{B}^{\mathcal{N}}$ and $\forall x_i \in \mathbf{X}$:

$$\pi_i^{\mathcal{N}} \left(b_i^{\mathcal{N}}, \mathbf{B}_{-i}^{\mathcal{N}}, x_i\right) - \tilde{\pi}_i \left(\tilde{b}_i, \mathbf{B}_{-i}^{\mathcal{N}}, x_i\right) \quad \geq \quad 0 \tag{6}$$

in which $\tilde{\pi}_i \left(\tilde{b}_i, \mathbf{B}_{-i}^{\mathcal{N}}, x_i\right)$ represents player i's surplus if he exploits the collusion with a bid $\tilde{b}_i \neq b_i^{\mathcal{N}}$ when the other plays according to the strategy $b^{\mathcal{N}}$. Given allocation ι, profits from upward and downward deviations take the following formulation:

$$\forall l > k: \ \tilde{\pi}_{\iota(k)} \left(\tilde{b}_{\iota(k)}, .\right) \quad = \quad \alpha_l \left(x_{\iota(k)} - \max\left\{b_{\iota(l+1)}^{\mathcal{N}}, 0\right\}\right) - \omega_{\iota(k)} \tag{7}$$

$$\forall s < k: \ \tilde{\pi}_{\iota(k)} \left(\tilde{b}_{\iota(k)}, .\right) \quad = \quad \alpha_s \left(x_{\iota(k)} - \max\left\{b_{\iota(s)}^{\mathcal{N}}, 0\right\}\right) - \omega_{\iota(k)} \tag{8}$$

and finally subject to individual rationality:

$$\pi_i^{\mathcal{N}}\left(b_i^{\mathcal{N}},\mathbf{B}_{-i}^{\mathcal{N}},x_i\right) \quad \geq \quad \pi_i\left(b_i,\mathbf{B}_{-i},x_i\right) \tag{9}$$

and the restriction that $b_1^{\mathcal{N}} \geq b_2^{\mathcal{N}} \geq b_3^{\mathcal{N}}$.

The incentive compatibility constraint (6) expresses the idea that once he receives his recommended bid from the centre (prior to playing in the auction), bidder i has the choice between obeying and adjusting his bid against what his competitors are asked to play at the auction[13]. If deviation occurs, the defector does not receive his individual compensation τ but still incurs his *supposed efficient* contribution as a *sunk cost*. This constraint requires that there does not exist a deviating bid $\tilde{b}$ that allocates bidder i any different position so that he is better off. The last constraint means that player i finds it profitable to join the *ring*.

Definition 1. A collusive mechanism $\boldsymbol{\zeta} = \left(\boldsymbol{\mu},\mathbf{p}^{\mathcal{N}}\right)$ is an equilibrium profile if it is (i) individually rational and (ii) incentive compatible.

In single unit auctions, a collusive device is said to implement an efficient outcome if the highest-valuing member of the collusion represents the cartel, wins the object and the ring is able to suppress internal competition. Such an outcome naturally maximizes social welfare as the good is allocated to the highest-valuing player. We will slightly modify this definition to make it coherent with the position auction context.

Definition 2 (*First-best outcome*)**.** A collusion in the position auction is an efficient mechanism if: (i) the final allocation is such that only the two-highest valuing members are active during the targeting auction, (ii) it suppresses competition from bidders not allocated to any position, (iii) it maximizes social welfare defined as the quantity[14] $\mathcal{W} = \sum_{j=1}^{2} \alpha_j x_{\iota(j)}$.

3 Benchmark: the non-cooperative outcome

In the present setting, we can easily compute the equilibrium strategy profile of the *GSP* following the work by Varian (2007), Edelman et al. (2007).

[13] Assuming they behave according to the recommendation.

[14] $\mathcal{W}$ can be thought of as a utilitarian welfare function which is obviously maximized whenever the condition $x_{\iota(1)} > x_{\iota(2)} > x_{\iota(3)}$ is satisfied. Therefore, an equilibrium ranking ι will be *efficient* only if players are assigned according to their indices. As we order bidders in decreasing order of their valuation this implies that the function ι has to be the identity mapping.

Proposition 0. *The following strategy profile* $\mathbf{b} = (b_1, b_2, b_3)$ *essentially characterized all the Nash equilibria of the static GSP auction with complete information:*

$$
\begin{aligned}
b_1 &\in \left[x_2 - \frac{\alpha_2}{\alpha_1}(x_2 - b_3)\,;\,\bar{x}\right] \\
b_2 &\in \left[\max\{x_3\,,\,b_3\}\,;\,x_1 - \frac{\alpha_2}{\alpha_1}(x_1 - b_3)\right] \\
b_3 &\in \left[\max\left\{0\,,\,x_1 - \frac{\alpha_1}{\alpha_2}\,(x_1 - b_2)\right\}\,;\,\min\{x_1\,,\,x_2\}\right]
\end{aligned}
\tag{10}
$$

The VCG-equivalent strategy profile $\mathbf{b}^v = (b_1^v, b_2^v, b_3^v)$ *is given by:*

$$
\begin{aligned}
b_1 &> b_2^v \\
b_2 &= x_2 - \frac{\alpha_2}{\alpha_1}\,(x_2 - b_3^v) \\
b_3 &= x_3
\end{aligned}
\tag{11}
$$

Notation 1. Two equilibria of the GSP in this framework reside at the boundaries of the Nash set we will refer to as *Lower-Nash Equilibrium* (LE) and *Upper-Nash Equilibrium* (UE). Each profile will be denoted respectively by $\mathbf{b}^l = \left(b_i^l\right)_{i=1,2,3}$ and $\mathbf{b}^u = (b_i^u)_{i=1,2,3}$. We will call the equilibrium achieved when restricting b_3 to be equal to x_3 the *Dom*-Nash equilibrium denoted by $\mathbf{b}^d = \left(b_i^d\right)_{i=1,2,3}$.

Each player can envision a multiplicity of best-responses to each other's equilibrium strategy. Equilibrium bids are bounded above and below by a combinations of the lowest bidder's bid and valuations of the player above and below him. Two things are worth noticing about the bid profile $\mathbf{b}$. First, does not rule out inefficient and non-assortative allocations. Actually, the only allocation ruled out is the one in which the highest-valuing advertiser wins no position and the lowest-valuing wins the top position[15]. Second, overbidding could be a candidate for a Nash equilibrium so that, it does not restrict attention to undominated strategy. The bid profile $\mathbf{b}^v$ is obtained using the *stability criterion* of *locally envy-freeness* of Edelman et al. (2007) and corresponds to the lowest point in the set of competitive prices among all the *locally envy-free* equilibria. Even if truth-telling is not an equilibrium strategy of the GSP auction, the outcome is strictly equivalent to what a VCG mechanism would have implemented. From a revenue perspective, any *symmetric* equilibrium (or *locally envy-free* equilibrium) induces a revenue at least equal to VCG revenues. As a result, if we denote by R^l and R^u the auctioneer's revenues achieved by $\boldsymbol{b}^l$ and $\boldsymbol{b}^u$ respectively we have that $R^l \leq R^v \leq R^u$.

[15]Indeed, if the highest-valuing player was to win no position then it should be the case that $b_3 \geq x_1 \geq x_3$ which cannot be an equilibrium profile. If now the lowest-valuing player was to now win the top position then it should be the case that $p_3 = b_1 > x_3$, implying a strict loss for him.

4 Results

4.1 Collusive equilibria

Consider that the *intermediary firm* acts as an incentiveless and credible banker, to whom bidders entrust their expected cash surplus from colluding *efficiently*. He computes individual contributions conditionally to the efficient Nash allocation deduced from the set (10). We assume that contributions are *allocation-independent,* computed *ex ante* and are thus not altered by potential deviations. Here, even if playing on their behalf is impossible or limited the *intermediary firm* will implement minimal but sufficient penalties to deter defection.

As in the *non-cooperative* framework, *ring* members can envision multiple best-responses against each other's equilibrium collusive strategy and may allow for asymmetric equilibria (see set (1) in appendix). For instance, it will still be an equilibrium if the highest-valuing member bids at his lower boundary and the second-highest valuing member bids at his upper one and the last player bids zero. However, the mechanism results in the highest-valuing *ring* members bidding at the auction stage and suppressing the bid of the low-valued member.

Proposition 1. *The collusive equilibrium with side-payments of the GSP auction with three players and two positions is characterized by an equilibrium bid profile* $\boldsymbol{\mu}_{\mathcal{N}} = \left(b_i^{\mathcal{N}}\right)_{i=1,2,3}$ *monotonically non-decreasing in* ε *and decreasing in* $\mathbf{b} = (b_i)_{i=1,2,3}$*. Moreover,* $\exists\delta > 0 : \forall\theta \geq \eta \wedge \forall\varepsilon \leq \delta$*, a constrained first-best is achieved and* $\forall\varepsilon > \delta$ *internal bidder's rivalry can no longer be constricted to a level below the valuation of the non-assigned member. The bid profile* $\boldsymbol{\mu}_{\mathcal{N}} = \left(b_i^{\mathcal{N}}\right)_{i=1,2,3}$ *is such that:*

$\forall\varepsilon \leq \delta \wedge \theta \geq \eta :$

$$\begin{aligned} b_1^{\mathcal{N}} &= x_3 - \frac{1-\varepsilon}{3\alpha_1}\left(w_1 + w_2\right) \\ b_2^{\mathcal{N}} &= \frac{1}{3\alpha_2 - \left(1-\varepsilon\right)\alpha_1}\left(3\alpha_2 x_3 - \left(1-\varepsilon\right)\sum_{i=1}^{2}\alpha_i b_{i+1}\right) \\ b_3^{\mathcal{N}} &= 0 \end{aligned} \tag{12}$$

$\forall\varepsilon \leq \delta \wedge \theta < \eta :$

$$\begin{aligned} b_1^{\mathcal{N}} &\in \left[b_2^{\mathcal{N}}, \bar{x}\right] \\ b_2^{\mathcal{N}} &= b_3^{\mathcal{N}} = 0 \end{aligned}$$

$$
\begin{aligned}
&\forall \varepsilon > \delta : \\
b_1^{\mathcal{N}} &\in \left[b_2^{\mathcal{N}}, \bar{x}\right] \\
b_2^{\mathcal{N}} &= \frac{\left((2+\varepsilon)\,\alpha_2 x_3 - (1-\varepsilon)\sum_{i=1}^{2}\alpha_i b_{i+1} - (1-\varepsilon)\,(\alpha_1-\alpha_2)\,x_1\right)}{(2+\varepsilon)\,\alpha_2 - (1-\varepsilon)\,2\alpha_1} \\
b_3^{\mathcal{N}} &= \frac{\left(x_1\,(\alpha_1-\alpha_2)\,((1-\varepsilon)\,\alpha_1 - 3\alpha_2) - (1-\varepsilon)\,(\alpha_1+\alpha_2)\sum_{i=1}^{2}\alpha_i b_{i+1} + (4-\varepsilon)\,\alpha_1\alpha_2 x_3\right)}{\alpha_2\,((2+\varepsilon)\,\alpha_2 - (1-\varepsilon)\,2\alpha_1)}
\end{aligned}
\tag{13}
$$

where

$$
\begin{aligned}
\delta &= \frac{(1+\theta)\,R - (1-\theta)\,(\alpha_1 - 3\alpha_2)\,x_1 - 4\alpha_2 x_3}{(1+\theta)\,R - (1-\theta)\,\alpha_1 x_1 - \alpha_2 x_3} \\
R &= \sum_{i=1}^{2}\alpha_i b_{i+1}
\end{aligned}
$$

Proof. See appendix (A.2). □

Proposition (1) asserts that a *ring* can achieve an outcome which is efficient according to definition (2). The highest-valuing player is allocated the top position and the second-valuing one the second position, while the lowest-valuing player refrains from bidding. There exists a whole bundle of fees so that, if the degree of substitutability between both positions is sufficiently high, the mechanism results in an outcome at which the *low-price* property prevails. In equilibrium, assigned *ring* members bid at the main auction a quantity strictly below the valuation of the non-assigned member, who refrains from bidding, which contrasts with the result found by Vorobeychik and Reeves (2008)[16].

The relation between the difference in the click-through rates between both position under coordination appears to be counterintuitive. Indeed, in the *non-cooperative* game, in order to stay optimal, Nash equilibrium bids are decreasing in the click ratio θ. Hence, if the substitutability between both positions becomes perfect, bidders should bid less as the need to outbid competitors to win the highest position decreases. However, when coordination is active, if the substitutability between positions increases, incentive compatibility constraints become stringent. The low-valued player's profit from cheating and aiming at the second position increases with θ. To see this, recall that incentives compatibility constraints for this member are given by the following relations:

$$
\begin{aligned}
IC_3 &: \tau \geq \alpha_1\left(x_3 - b_1^{\mathcal{N}}\right) \\
IC_3' &: \tau \geq \alpha_2\left(x_3 - b_2^{\mathcal{N}}\right)
\end{aligned}
$$

[16] In their model, a collusive equilibrium profile emerges as a process of *non-cooperative* behaviours when considering the repeated *GSP* auction. It is an equilibrium to bid, for each player within the position set, an *epsilon* above the valuation of the first not being assigned a position and all the others play truthfully. If one deviates from the collusive strategy then all trigger the minimum revenue *symmetric* Nash equilibrium strategies. Here, coordination is done without the help of an *intermediary*.

His utility payoff function from deviating for position one is independent of θ. However, it is strictly increasing in θ for the second one. As a result, the centre is constrained to released the second-highest valuing member in order to maintain the incentives aligned and the ranking, which in turn makes the internal competition difficult to contain.

Remark 2. The click ratio constraints η and ρ are meaningless when the centre sets $\varepsilon > \delta$. Indeed, the threshold couple (η, ρ) makes no sense since (i) relation (13) holds $\forall \varepsilon \in [0, \delta]$. Thus, the second-valuing member's bid is strictly positive once ε reaches the value δ and is, thus, also strictly positive for $\varepsilon = \delta + \sigma$. Then, (ii) $\forall \varepsilon \leq \delta$ if $\theta > \eta$ his bid is also strictly positive. Finally, once $\varepsilon > \delta$ we have that

$$\frac{\partial \left(\frac{(1-\varepsilon)(\alpha_1 b_2 + \alpha_2 b_3) + (1-\varepsilon)(\alpha_1 - \alpha_2) x_1 - (2-\varepsilon)\alpha_2 x_3}{(1-\varepsilon) 2\alpha_1 - (2-\varepsilon)\alpha_2} \right)}{\partial \varepsilon} \quad \geq \quad 0$$

which implies that $b_2^{\mathcal{N}} > 0$ independently of the value of θ and that $b_3^{\mathcal{N}}$ is also necessarily positive. Therefore $\forall \varepsilon > \delta$ the thresholds η and ρ become unbinding.

Note that δ is a non-decreasing function of the *non-cooperative* bid profile $\boldsymbol{b}^g = (b_i^g)_{i=1,2,3}$ with $g = l, v, u$. It implies a more stringent condition over individual contributions ω_i if the Nash bids are assumed to be played according to $\mathbf{b}^l$. A small increase in the fee makes the low-valued member to set $b_3^{\mathcal{N}} > 0$, which mechanically increases within-competition. The *IC* constraints become binding and members' incentives reverse. As a result, the *first-best outcome* property can no longer be maintained if the centre is to become too greedy as regards the surplus appropriation. It causes the collusion to be crushed from the inside. Thus, using the words of McAfee and McMillan (1992), because of the existence of a bargaining power from the intermediary firm, the ring in this framework "*contains the seeds of its own destruction*"[17].

We give a representation of the equilibrium strategies in figure (1). The bid profile $\boldsymbol{\mu}_{\mathcal{N}} = \left(b_2^{\mathcal{N}}, b_3^{\mathcal{N}}\right)$ increases in the fee and both functions are non-differentiable at threshold δ. We can see that from δ within-competition strictly increases as the lowest-valuing member's bid becomes strictly positive. The monotonicity of the equilibrium bidding functions is a straight implication of two correlated effects. (i) Once the centre decides to set its fee ε to a higher level, bidding functions consequently increase, which mechanically entails a decrease in the available surplus. In return, (ii) the centre will need to break incentives to defect for the non-assigned member, which implies an increase in the overall collusive bid functions. To see why, recall that individual payoff utility function is equal to:

$$\pi_i^{\mathcal{N}} \left(x_i; b_i^{\mathcal{N}} \right) \quad = \quad \alpha_k \left(x_k - b_{k+1}^{\mathcal{N}} \right) - \left(\alpha_k b_{k+1} - \alpha_k b_{k+1}^{\mathcal{N}} \right) + \frac{1-\varepsilon}{3} \sum_{j=1}^{2} \alpha_j b_{j+1} - \frac{1-\varepsilon}{3} \sum_{j=1}^{2} \alpha_j b_{j+1}^{\mathcal{N}}$$

[17] In McAfee and McMillan (1992) this is due to the issue of profit attractiveness from the collusion implying an entry problem.

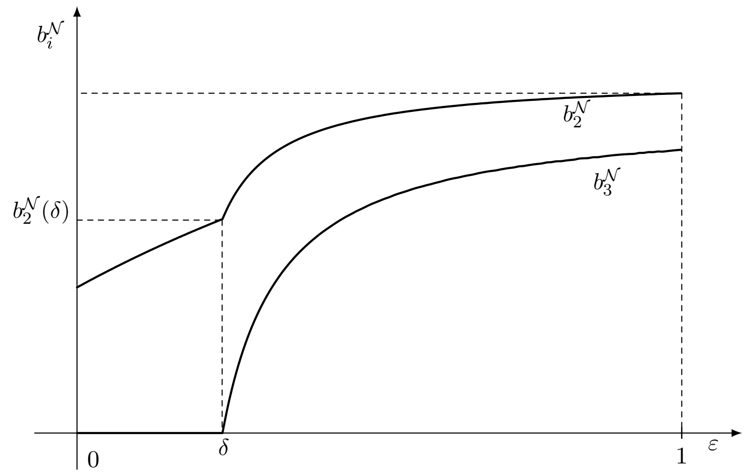


Figure 1: Collusive equilibrium bidding functions.

it is thus obvious to see that:

$$\frac{\partial^2 \pi_i}{\partial \varepsilon \partial b_i^{\mathcal{N}}} > 0$$

This means that, the marginal loss related to an increase in the fee is compensated by a marginal increase in the coordinated bid. This compensation is more likely to be high as the latter bid is high. Thus, a greedy behavior from the *intermediary firm* gives rise to more aggressive behaviours from members in order to maintain incentive compatibility and to compensate the downward shifting of the collusive gain in terms of individual contributions. These two correlated effects corroborate the intuitive idea that the bargaining power of the centre has a substantial pervasive effect upon the durability and the sustainability of the collusion.

Finally, another insight offered by propositions (1) relies on the question of the sustainability of the *non-cooperative* mechanism as a limit case of any collusive mechanism. We know by proposition (0) that the complexity of the *GSP* auction gives rise to a multiplicity of Nash equilibria. In our framework, the *intermediary firm* has the power to push the equilibrium prices upward so that bidders reverse to *non-cooperative* behaviours. Consider for a while that the *intermediary firm* endorses the role of an exogenous equilibrium "perturbator" (as a mediator in Ashlagi et al. (2009), Monderer and Tennenholtz (2009)); up to which point can this be done?

Corollary 1. *The* non-cooperative *equilibrium is a sustainable collusive outcome with a uniform-ε redistribution rule. Coordinated payoff utility functions are bounded above by the payoff utility functions in the corresponding non-cooperative equilibrium bid profile* $\mathbf{b}^l = \left(b_i^l\right)_{i=1,2,3}$.

Proof. See appendix (A.3). □

The simplest *IC* collusive mechanism which is *always feasible* turns out to be the *non-cooperative* mechanism in which players set their optimal bids consistently with the Nash equilibrium criterion.

By incrementally increasing its expropriation capability the intermediary firm is able to evict all other equilibria, so that the *lowest* Nash equilibrium outcome $\mathbf{b}^l = \left(b_i^l\right)_{i=1,2,3}$ becomes the unique ending point. This implies that individual payoffs from coordination are no higher than in the corresponding equilibrium payoffs in $\mathbf{b}^l = \left(b_i^l\right)_{i=1,2,3}$.

Example 1. Let the set of valuation be $\boldsymbol{x} = (5, 4, 3)$ and the value of both click-through rates be $\boldsymbol{\alpha} = (10, 8)$. Suppose that as an outside option, the *intermediary firm* conjecture that bidders will play according to the bid profile $\boldsymbol{b}^u = (b_1^u, b_2^u, b_3^u)$ which is so that $b_3^u = 4, b_2^u = 4.2$ and $b_1^u = 5$. In this environment $\delta \simeq 0.51$ and $\eta = 0.84$ so $\theta < \eta$. The equilibrium proposal $\boldsymbol{\mu}_{\mathcal{N}} = \left(b_1^{\mathcal{N}}, b_2^{\mathcal{N}}, b_3^{\mathcal{N}}\right)$ for $\varepsilon = 0$ is given by $b_3^{\mathcal{N}} = 0, b_2^{\mathcal{N}} = 0$ and $b_1^{\mathcal{N}} > 0$.

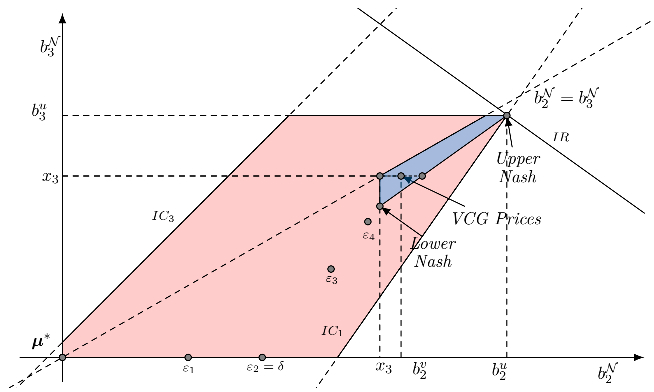


Figure 2: Set of collusive bids in light red with $\varepsilon_0 = 0, \varepsilon_1 = 0.3, \varepsilon_2 = \delta, \varepsilon_3 = 0.6$ and $\varepsilon_4 = 0.9$.

The mechanism results in both player 2 et 3 to refrain from bidding, which results in a suboptimal allocation (both will be randomly assigned position 2). The *non-cooperative* game results in a revenue of $R = \alpha_1 b_2 + \alpha_2 b_3 = 74$ and $R_{\boldsymbol{\mu}_{\mathcal{N}}} = 0$. As said by the corollary, the collusive outcome converge to $\boldsymbol{b}^l$ as $\varepsilon \mapsto 1$. At this point, $\boldsymbol{\mu}_{\mathcal{N}} = \left(b_1^{\mathcal{N}}, b_2^{\mathcal{N}}, b_3^{\mathcal{N}}\right) = \left(b_2^{\mathcal{N}} + \sigma, 3, 2.5\right)$ and generates a revenue $R_{\boldsymbol{\mu}_{\mathcal{N}}} = 50$. Consider that the outside bid profile is set to $\boldsymbol{b}^l = \left(b_1 > b_2, x_3, x_1 - \frac{\alpha_1}{\alpha_2}(x_1 - x_3)\right)$, then $R = \alpha_1 b_2 + \alpha_2 b_3 = 50$ and $R_{\boldsymbol{\mu}_{\mathcal{N}}} = 50$ if $\varepsilon = 1$.

4.2 Revenues: compatibility between the auctioneer and the intermediary firm

Besides, we learnt that a profitable coordination can be done even if the centre increases its monopoly power over *ring* members. It still keeps track of the payoff dominance of the collusion over the *non-cooperative* game and does not alter the allocative efficiency of the mechanism. However, the presence of an *intermediary firm* does affect the auctioneer's revenues. In a nutshell, introducing the ability to extract a positive fee creates some tensions between expropriation of surplus, constriction of bidders' rivalry and a *low-price* outcome.

Proposition 2. *The* centre*'s revenue is non-monotonic in ε. An optimal incentive-compatible fee that an intermediary firm can set is the point $\varepsilon^* = \delta$.*

Proof. See appendix (A.4). □

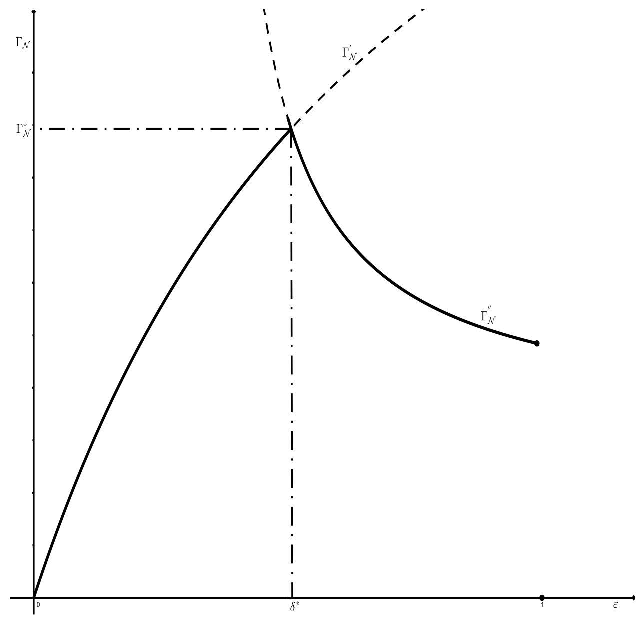


Figure 3: The intermediary firm profits as a function of ε

The profits functions are depicted in figure (3). While bidding functions are monotonically increasing we observe that the *intermediary firm* is able to increase its profits. The increase in the fees sufficiently compensates for the one in the collusive price. As a result, biasing the redistribution of the surplus becomes a *plausible and valuable* strategy as it can be done without deterring the collusive incentives. The threshold δ is the critical point at which ε becomes unsustainable and from which a greedy attitude becomes pervasive. The loss in the bidders' surplus needs to be compensated for by an increase in the respective collusive bids, which implies a shift in the *intermediary firm*'s profits. A rational behaviour that mechanically implies an increase of the internal competition level, so that the collusion can no longer be constrained to an efficient level according to definition (2).

In fact, the latter is somehow a "common sense" observation as anyone could have expected this aggressive result. The level of expropriation from the centre makes the incentive conditions more stringent for the bidders to be self-interested in the cartel formation. Interestingly, such a greedy behavior from the *intermediary firm* does not induce an antinomic relationship with the auctioneer and may be beneficial to him as $\boldsymbol{\mu}_{\mathcal{N}} = \left(b_i^{\mathcal{N}}\right)_{i=1,2,3}$ is strictly increasing in ε.

Consider now that the auctioneer has the power to affect the click-through rate couple $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$. A strong assumption as the α_is are the result of customers' searching strategies, but let us abstract away this feature and let us suppose that, indeed, the auctioneer was to set them arbitrarily. We claim the following:

Claim 1. An auctioneer cannot use the ratio θ in order to induce levels of revenue higher than those of the lowest Nash equilibrium.

Proof. See appendix (A.5). □

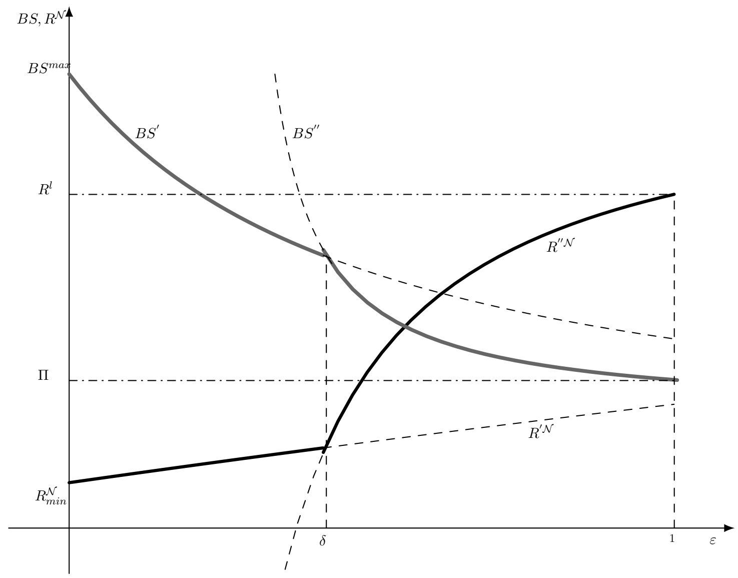


Figure 4: The auctioneer's revenue $R^{\mathcal{N}}$ and bidders' surplus BS as a function of ε

Together with the fact that $\lim_{\varepsilon \mapsto 1} b_i^{\mathcal{N}} = b_i^l$, proposition (1), which says that the bid profile $\boldsymbol{\mu}_{\mathcal{N}}$ is decreasing in the bid profile $\mathbf{b}$ and proposition (2), which says that it is optimal for the *intermediary firm* to set $\varepsilon = \delta$, we can state the following:

Proposition 3. *The auctioneer's revenue is monotonically increasing in ε and the maximum quantity of surplus it can extract is no higher than in the corresponding non-cooperative equilibrium bid profile $\mathbf{b}^l$.*

An illustrative representation of the auctioneer's revenue $R^{\mathcal{N}}$ (the black lines) as a function of ε against the overall bidders' surplus BS (the grey lines) is given in figure (4). If $\varepsilon = 0$ the overall surplus will be entirely divided between members and the auctioneer; nothing is expropriated by it. He gets the minimal revenue that the presence of the efficient ring generates and the members get the maximal surplus they can keep from him. If the *intermediary firm* sets the maximal fee (if it intends to act as a sort of *non-cooperative* equilibrium decentralizer), i.e., $\varepsilon = 1$, $\boldsymbol{\mu}_{\mathcal{N}}$ converges to $\mathbf{b}^l$, so does the auctioneer's revenue[18]. Recall that, in order to maintain a sufficient degree of spoils, members of the ring need to push forward the equilibrium prices which is mechanically beneficial to the seller despite the fact that this surplus now has to be divided between the players, the seller and the centre. In the figure we denote by $\Pi = \sum_{i=1}^{3} \pi_i$ the overall surplus achieved under the outcome $\mathbf{b}^l$, and by R^l the maximal revenue to the auctioneer.

In all cases the auctioneer benefits from of greedy behaviour from the *intermediary firm* and is bounded above R^l achieved under $\mathbf{b}^l$. A positive relation which entails an interesting issue. Given the impossibility for a auctioneer to artificially and indirectly drive up the collusive bids by claim (1) and given that he strictly benefits from an increase in the fee, we could naturally guess that it could be in his interest to manage a tacit agreement with the *intermediary firm*. Indeed, it is not in

[18]See figure (7) for a numerical illustration in which the auctioneer's revenue is depicted for each Nash extremum.

the interest of the latter agent to set a fee $\varepsilon > \delta$. As a result, the auctioneer could handle sufficient compensations in order to make this strategy incentives compatible for the *intermediary firm*. In other words, this positive relationship suggests the possible and intuitive issue of a corruption device between the auctioneer and the centre. Such analysis of a corrupted auctioneer is widely beyond the scope of the present work (see for instance works by Menezes and Monteiro (2006), Lengwiler and Wolfstetter (2010)).

4.3 Substitutability between positions and competition

Finally, we compare the link between the click ratio θ as a measure of the substitutability degree of positions and the *demand-reduction* phenomenon observed in the *GSP* auction and more generally in multi-object auctions. Bid shading occurs in situations in which players' bids affect the price they have to pay at the end of the auction which relies on the assumption that bidders have multi-unit demand. As a result, it should vanish whenever the same player only asks for one object (one position) at the most.

In the environment of the *GSP* auction, advertisers can ask for only one position. Notice the special feature in the competition nature and equilibrium predictions suggested by the bid profile $\mathbf{b}^v$. In equilibrium, it is optimal for each player to bid strictly below their own valuation (except for the non-assigned one). Actually, this phenomenon can be deepened or crushed with the differentiation in clicks between positions. This enlightens us as to the existence of a tight link between *ctr* and the competition nature in the *GSP* auction.

Remark 3. The relation between the ratio $\theta = \frac{\alpha_2}{\alpha_1}$, the equilibrium bids and payments is summarized in the following observations:

(*i*) As $\theta \mapsto 0$, bid shading vanishes. The game converges essentially to a standard second-price auction.

(*ii*) As $\theta \mapsto 1$, bid shading increases. The game essentially converges to a *Bertrand* competition.

Take the set of Nash equilibrium bids in (15) and observe that the bids are convex combinations weighted by the ratio θ. Without loss of generality, let us consider the refinement that, in the *non-cooperative* play the non-assigned bidder (bidder 3) will use a dominant strategy, i.e., $b_3 = x_3$.

$$\begin{aligned} b_1 &\in \left[x_2 - \frac{\alpha_2}{\alpha_1}(x_2 - x_3)\,;\,\bar{x}\right] \\ b_2 &\in \left[x_3\,;\,x_1 - \frac{\alpha_2}{\alpha_1}(x_1 - x_3)\right] \\ b_3 &= x_3 \end{aligned} \tag{14}$$

It is now obvious to see that when $\theta = 0$, i.e., the first position becomes the only object that is worth winning (the most attractive one), In this case, bids become bounded by the next-to-top valuation and the next-to-bottom one.

$$\begin{aligned} b_1 &\in [x_2\,;\,\bar{x}] \\ b_2 &\in [x_3\,;\,x_1] \\ b_3 &= x_3 \end{aligned} \tag{15}$$

The link is even tighter at the *symmetric* equilibrium in which, indeed, bidding one's valuation becomes the only possible bid as is the case in standard second-price auctions.

Now, for (*ii*), note that, as $\theta = 1$, both positions becomes substitutes and the mid-valuing player's equilibrium bid becomes equal to $\max\{b_3, x_3\}$. There is no opportunity for the highest-valuing player to undercut him and the lowest-valuing advertisers set the market-clearing price to his private valuation.

$$\begin{aligned} b_1 &\in [x_3\,;\,\bar{x}] \\ b_2 &= x_3 \\ b_3 &= x_3 \end{aligned} \tag{16}$$

This situation highlights the analogy with a Bertrand competition and the *demand-reduction* phenomenon well-known in standard multi-object auctions[19]. The lowest-valuing bidder's private valuation determines the shape of the market-clearing price. Nevertheless, there is still an opportunity for the auctioneer by making the top position the only one worth having to destroy incentives to shade in the *non-cooperative* play.

However, there are no clear tendencies for the link between θ and the competition nature with an active cartel. If the substitutability between position was to decrease, that is $(\alpha_1 - \alpha_2)$ increase, then if the *intermediary firm* behaves as in proposition (1) and set $\varepsilon = \delta$ the coordinate bids of player 2 equals zero and the link reverses. As a result, we nest the standard result of collusion in second-price auctions with multi-unit objects and the bid shading is exacerbate[20]. However, if $\theta = 1$ which means that $(\alpha_1 - \alpha_2)$ increase, then,

$$\begin{aligned} b_2^{\mathcal{N}} &= \frac{3x_3 - (1-\varepsilon)(b_2 + b_3)}{2+\varepsilon} \\ b_3^{\mathcal{N}} &= 0 \end{aligned}$$

which depends on the bid profile $\mathbf{b}$. If we assume that the centre seeks to maximize the bidders surplus, then he coordinates the outside option towards $\mathbf{b}^l = \left(b_2^l = x_3, b_3^l = x_1 - \frac{\alpha_1}{\alpha_2}(x_1 - x_3)\right)$ and

[19] See for instance Engelbrecht-Wiggans and Kahn (1998), Ausubel et al. (2014).

[20] e.g., Graham and Marshall (1987), McAfee and McMillan (1992), Milgrom (2000), Brusco and Lopomo (2002)

when $\varepsilon = \delta$ we get,

$$\begin{aligned} b_2^{\mathcal{N}} &= \frac{1}{2} x_3 \\ b_3^{\mathcal{N}} &= 0 \end{aligned}$$

For $\varepsilon > \delta$, then $b_2^{\mathcal{N}} = x_3 = b_3^{\mathcal{N}}$. Thus, if $(\alpha_1 - \alpha_2)$ increase, then for $\varepsilon = \delta$ the bid of player 2 equals half of the valuation of the lowest-valued bidder whereas $\forall \varepsilon > \delta$ then $b_2^{\mathcal{N}}$ increase to x_3. As a result, coordinated bids increase in θ and in the degree of subtitutability between positions.

4.4 Extension: symmetric collusive equilibrium

In this section we shall show that, if we scale the analysis over the spirit of standard competitive equilibrium analysis by using the LEF criterion in the present framework, then there does not exist a profitable way to collude except to achieve the *VCG* outcome. That is to say, if deviations are managed by members according to the standard argument of *local indifference* then the equivalence with the *Vickrey-Clarkes-Groves* solution is restored and is furthermore unique.

Let us recall that the LEF criterion implies a shift in the *IC* conditions of a Nash equilibrium so that upward and downward deviations become symmetric. Then, in the spirit of Edelman et al. (2007), in order to implement a stable equilibrium, bidders need to be *locally indifferent.* That is, a bidder assigned a position k when contemplating a move to position $k-1$ has to expect to pay the same price as the player in this position and to be *indifferent* between both assignments. Now what would be the effect if we bring such a *stability condition* to the collusive game, selecting among the multiplicity of collusive equilibria the one who respects this criterion? In other words, let us consider the LEF criterion as a *stability condition* instead of the expression of a *Walrasian tatônnement* process (as in Börgers et al. (2013)), the latter being necessarily related to a *non-cooperative* analysis.

Still assume that, in the case of a deviation from the collusive agreement, the defector pays his individual contribution computed *before the targeting auction starts.* That is individual contributions are computed unconditionally on deviations and based on an assortative assignment. In addition, still assume that the centre makes his transfer payment after the main auction. The *incentive compatibility* conditions are now given by the following relations: $\forall i = 1, 2, 3$, $\forall b_{\iota(i)}^{\mathcal{N}} \in \mathbf{B}^{\mathcal{N}}$ and $\forall x_i \in \mathbf{X}$:

$$\pi_{\iota(k)}^{\mathcal{N}} \left(b_{\iota(k)}^{\mathcal{N}}, \mathbf{B}_{-i}^{\mathcal{N}}, x_{\iota(k)} \right) - \tilde{\pi}_{\iota(k)} \left(\tilde{b}_{\iota(k)}, \mathbf{B}_{-i}^{\mathcal{N}}, x_i \right) \geq 0 \tag{17}$$

where $\forall k \in \mathcal{K}$:

$$\tilde{\pi}_{\iota(k)} \left(\tilde{b}_{\iota(k)}, . \right) = \alpha_{k-1} \left(x_{\iota(k)} - b_{\iota(k)}^{\mathcal{N}} \right) - \left(\alpha_k b_{\iota(k+1)} - \alpha_k b_{\iota(k+1)}^{\mathcal{N}} \right) \tag{18}$$

This condition says that, when player $\iota(k)$ assigned to position k intends to defect the collusive agreement and contemplates position $k-1$ then he expects to pay the same price as player $\iota(k-1)$ assigned to this position. Hence, we give the mechanism a semblance of symmetry in the collusive equilibrium conditions and the latter results in each members bidding the *non-cooperative symmetric* equilibrium prices.

Proposition 4. *Under the locally envy-free condition and a* uniform-ε *redistribution a cartel with three players can do no better than to achieve the* non-cooperative *outcome. That is:* $\boldsymbol{\mu}^*_{\mathcal{N}} = \left(b_1^{\mathcal{N}} > b_2^{\mathcal{N}} \,;\, \frac{P_1^v}{\alpha_1} \,;\, x_3\right) = \mathbf{b}^*$.

It is interesting to note that bidding functions are now completely independent of the fee that the intermediary firm might set. This proposition also highlights the robustness of Varian (2007), Edelman et al. (2007) stability criteria to any collusive agreements implying a lump-sum transfer of the spoils. Such observation is obviously made with respect to our model.

5 Concluding remarks

We have analysed a situation in which a third party enters the *GSP* auction game and emulates an explicit form of coordination among bidders who compete for the same set of positions. Explicit collusion here means the ability to implement side contracts in order to achieve a better outcome than the lowest *non-cooperative* one. Collusion in the present model works as follows: each bidder who wants to participate in the ring is asked for an individual contribution by giving a monetary amount to the organizer. In exchange, at the end of the auction they get an equal share of the collusive surplus. The centre makes a lump sum transfer to each member.

We find that an efficient collusion with the *low revenue* property is sustainable in equilibrium. Internal competition is constricted to a level strictly below the valuation of the lowest-valuing member (the one who is not assigned to any position). Such an outcome is implemented even if the third party levies a positive fee at a threshold which entails the highest level of payoff to him. Among the multiplicity of collusive equilibria, we have also found that, the *lowest* Nash equilibrium constitutes a limit case to any coordination. This equilibrium forms an upper boundary on the bidders' surplus. This fact also sheds light on the maximal revenue that can be extracted by the seller. In absence of the reserve price tool the most that it can achieve is the revenue generated at this upper boundary.

In contrast to the *non-cooperative* game we have observed a reverse relation between the competition nature and the degree of substitutability between positions. We have provided elements showing that the degree of shading cannot be lower than in the lowest *non-cooperative* equilibrium. In the mean time, based on observed market structures we propose another kind of justification for the choice of the *VCG*-equivalent outcome. We find that it is implemented as a unique equilibrium as the result of explicit coordinated behaviours between bidders. To our concern, this gives an interesting consistence to the justification of the *VCG*-equivalent outcome. It is no longer based solely on reasonable guesses and strengthens the *symmetric stability criterion*.

Our analysis rests on the assumption that valuations are common knowledge and that the *intermediary firm* does not suffer an *adverse selection* issue when implementing the collusive device. Yet, because of tractability issues there have been few studies examining *GSP* auction with incomplete information (e.g., Gomes and Sweeney (2014)). Therefore, we find that an adequate understanding of how collusion with side payments works with complete information is necessary before further studying collusion in position auctions with incomplete information.

Another limitation to the present analysis stems from the equal contribution assumption. It would be interesting to allow the payments from the centre to the member to be conditional on the importance that each player has within the coalition. For instance, a redistribution as a function of the market power of each firm on the product market. Finally, it would be reasonable to consider that advertisers have different informations in hand, which could modify their market perception and their valuation for a click. Usually, they do have strictly more information than the seller about customers' purchasing behaviour on their own market. For instance, individual valuations should not be considered as monotonically decreasing across positions. Hence, this strong information asymmetry should be taken into account in the model as the latter is highly valuable information and would clearly affect the cartel sustainability

A Appendix

A.1 Proof of proposition (0)

An equilibrium outcome will be supported by a vector $\mathbf{b} = (b_1, b_2, b_3)$ of equilibrium bids if the following holds:

$$\begin{aligned}
\alpha_1 (x_1 - b_2) &\geq \alpha_2 (x_1 - b_3) \\
\alpha_2 (x_2 - b_3) &\geq \alpha_1 (x_2 - b_1) \\
\alpha_1 (x_1 - b_2) &\geq 0 \\
\alpha_2 (x_2 - b_3) &\geq 0 \\
0 &\geq \alpha_2 (x_3 - b_2) \\
0 &\geq \alpha_1 (x_3 - b_1) \\
b_1 \geq b_2 &\geq b_3 = 0
\end{aligned} \tag{19}$$

The first two inequalities are the incentive constraints for player 1 and 2 to not deviate for position 2 and 1 respectively. The second two inequalities denotes individual rationality constraint: a bidder if assigned cannot profitably deviate to win no position. The last inequalities express the non-incentives for player 3 to deviate and win any position and the monotonicity conditions. Rearranging inequalities results in the desired set. Note that the shape of prices modifies as upward deviation occurs, which is not in the spirit of usual competitive equilibrium analysis. Players can influence the price they pay at the end and are not taken. The Nash stability criterion is thus asymmetric.

Following Varian (2007), Edelman et al. (2007), the strategy profile $\mathbf{b} = (b_1, b_2, b_3)$ will be a *locally envy-free* equilibrium or a *symmetric* equilibrium if the following relation holds:

$$\alpha_k x_k - p_k \quad \geq \quad \alpha_{k-1} x_k - p_{k-1} \quad k = 1, 2, 3$$

with $p_k = \alpha_k b_{k+1}$.

No player has an incentive to deviate and to bid for the position just above him. More formally the player assigned to position k has to be *locally indifferent* between winning position $k-1$, paying his own bid and position k paying the next-highest bid. This implies that,

$$\begin{aligned} \alpha_k x_k - p_k &= \alpha_{k-1} x_k - p_{k-1} \\ p_k &= \alpha_i b_{i+1} = \sum_{k=i+1}^{3} x_k (\alpha_{k-1} - \alpha_k) \end{aligned}$$

Hence $\boldsymbol{b}^v$ is defined by,

$$\begin{aligned} b_1^v &> b_2^v \\ b_2^v &= x_2 - \frac{\alpha_2}{\alpha_1} (x_2 - x_3) \\ b_3^v &= x_3 \end{aligned}$$

A.2 Proof of proposition 1

We want to construct a collusive equilibrium bid profile which is compatible with incentive compatibility constraints defined in (7) and (8) and with the individual rationality constraint defined by the relation (9). We set $\theta = \frac{\alpha_2}{\alpha_1}$.

From player 2's incentive compatibility constraint, he will not deviate for position 1 if the following relation is satisfied:

$$\alpha_2 \left(x_2 - b_3^{\mathcal{N}} \right) - \omega_2 + \frac{1 - \varepsilon}{3} \Pi_{\mathcal{N}} \quad \geq \quad \alpha_1 \left(x_2 - b_1^{\mathcal{N}} \right) - \omega_2$$

This relation gives the following conditions:

$$\begin{aligned} b_3^{\mathcal{N}} &\leq \frac{1}{(4 - \varepsilon)\, \alpha_2} \left((1 - \varepsilon) \sum_{i=1}^{2} p_i - (1 - \varepsilon)\, \alpha_1 b_2^{\mathcal{N}} + 3 \alpha_1 b_1^{\mathcal{N}} - (\alpha_1 - \alpha_2)\, 3 x_1 \right) \\ b_1^{\mathcal{N}} &\geq \frac{1}{3 \alpha_1} \left((\alpha_1 - \alpha_2)\, 3 x_2 - (1 - \varepsilon) \sum_{i=1}^{2} p_i - (1 - \varepsilon)\, \alpha_1 b_2^{\mathcal{N}} + (4 - \varepsilon)\, \alpha_2 b_3^{\mathcal{N}} \right) \end{aligned}$$

giving the first lower boundary for player 1 and an upper one for player 3.

Now from incentive compatibility constraint of player 1, if he does not want to swap his position for position 2 then it should be the case that:

$$\alpha_1\left(x_1-b_2^{\mathcal{N}}\right)-\omega_1+\frac{1-\varepsilon}{3}\Pi_{\mathcal{N}} \;\;\geq\;\; \alpha_2\left(x_1-b_3^{\mathcal{N}}\right)-\omega_1$$

which gives the following relations:

$$\begin{aligned}
b_2^{\mathcal{N}} &\leq \frac{1}{(4-\varepsilon)\,\alpha_1}\left((\alpha_1-\alpha_2)\,3x_1+\sum_{i=1}^{2}p_i+(2+\varepsilon)\,\alpha_2 b_3^{\mathcal{N}}\right)\\
b_3^{\mathcal{N}} &\geq \frac{1}{(2+\varepsilon)\,\alpha_2}\left((4-\varepsilon)\,\alpha_1 b_2^{\mathcal{N}}-\sum_{i=1}^{2}p_i-(\alpha_1-\alpha_2)\,3x_1\right)
\end{aligned}$$

the first upper boundary for player 2 and the lower one for player 3.

Finally, looking at incentive compatibility constraint for player 3 we get the following:

$$\begin{aligned}
\frac{1-\varepsilon}{3}\Pi_{\mathcal{N}} &\geq \alpha_1\left(x_3-b_1^{\mathcal{N}}\right)\\
\frac{1-\varepsilon}{3}\Pi_{\mathcal{N}} &\geq \alpha_2\left(x_3-b_2^{\mathcal{N}}\right)
\end{aligned}$$

giving respectively:

$$\begin{aligned}
b_1^{\mathcal{N}} &\geq x_3-\frac{1}{3\alpha_1}\left((1-\varepsilon)\sum_{i=1}^{2}w_i\right)\\
b_2^{\mathcal{N}} &\geq \frac{1}{3\alpha_2-(1-\varepsilon)\,\alpha_1}\left(3\alpha_2 x_3-(1-\varepsilon)\sum_{i=1}^{2}p_i+(1-\varepsilon)\,\alpha_2 b_3^{\mathcal{N}}\right)
\end{aligned}$$

the second lower boundary for player 1 and 2.

The participation constraint equals,

$$\sum_{i=1}^{2}p_i \;\;\geq\;\; \sum_{i=1}^{2}\alpha_i b_{i+1}^{\mathcal{N}}$$

which implies $b_2^{\mathcal{N}}\leq b_2$ and $b_3^{\mathcal{N}}\leq b_3$.

The above inequalities results in the following equilibrium strategy profile $\boldsymbol{\mu}_{\mathcal{N}}=\left(b_i^{\mathcal{N}}\right)_{i=1,2,3}$:

Set of collusive bids 1.

$$\begin{aligned}
b_1^{\mathcal{N}} &\in \left[\max\left\{x_3-\frac{(1-\varepsilon)}{3\alpha_1}\sum_{i=1}^{2}w_i\,;\,\frac{1}{3\alpha_1}\left((\alpha_1-\alpha_2)\,3x_2-(1-\varepsilon)\sum_{i=1}^{2}p_i-(1-\varepsilon)\,\alpha_1 b_2^{\mathcal{N}}+(4-\varepsilon)\,\alpha_2 b_3^{\mathcal{N}}\right)\right\}\,;\,\bar{x}\right]\\
b_2^{\mathcal{N}} &\in \left[\max\left\{0\,;\,\frac{1}{3\alpha_2-(1-\varepsilon)\,\alpha_1}\left(3\alpha_2 x_3-(1-\varepsilon)\sum_{i=1}^{2}p_i+(1-\varepsilon)\,\alpha_2 b_3^{\mathcal{N}}\right)\right\}\,;\,\mathcal{A}\right]\\
b_3^{\mathcal{N}} &\in \left[\max\left\{0\,;\,\frac{1}{(2+\varepsilon)\,\alpha_2}\left((4-\varepsilon)\,\alpha_1 b_2^{\mathcal{N}}-(1-\varepsilon)\sum_{i=1}^{2}p_i-(\alpha_1-\alpha_2)\,3x_1\right)\right\}\,;\,\mathcal{B}\right]
\end{aligned} \tag{20}$$

with

$$\begin{aligned}
\mathcal{A} &= \frac{1}{(4-\varepsilon)\,\alpha_1}\left((\alpha_1-\alpha_2)\,3x_1+\sum_{i=1}^{2}p_i+(2+\varepsilon)\,\alpha_2 b_3^{\mathcal{N}}\right)\\
\mathcal{B} &= \frac{1}{(4-\varepsilon)\,\alpha_2}\left((1-\varepsilon)\sum_{i=1}^{2}p_i-(1-\varepsilon)\,\alpha_1 b_2^{\mathcal{N}}+3\alpha_1 b_1^{\mathcal{N}}-(\alpha_1-\alpha_2)\,3x_1\right)
\end{aligned}$$

We will use the following claim which gives us a tool in order to discriminate among the set of compatible collusive bids:

Claim 2. A sufficient condition for $b_2^{\mathcal{N}}>0$ is $\theta=\frac{\alpha_2}{\alpha_1}\geq\frac{(1-\varepsilon)b_2}{3x_3-(1-\varepsilon)b_3}$.

Proof. Set $b_1^{\mathcal{N}}$ so that neither player 2 nor player 3 have incentives to deviate for the top position. The incentive compatibility conditions are:

$$\begin{aligned}
(1-\varepsilon)\,(\alpha_1 b_2+\alpha_2 b_3)-3\alpha_2 x_3-(1-\varepsilon)\,\alpha_2 b_3^{\mathcal{N}}+(3\alpha_2-(1-\varepsilon)\,\alpha_1)\,b_2^{\mathcal{N}} &\geq 0\ IC_3\\
3\alpha_2 x_2+(1-\varepsilon)\,(\alpha_1 b_2+\alpha_2 b_3)-(4-\varepsilon)\,\alpha_2 b_3^{\mathcal{N}}+(1-\varepsilon)\,\alpha_1 b_2^{\mathcal{N}} &\geq 0\ IC_2\\
3\alpha_1 x_1+(1-\varepsilon)\,(\alpha_1 b_2+\alpha_2 b_3)-(1-\varepsilon)\,\alpha_2 b_3^{\mathcal{N}}+(4-\varepsilon)\,\alpha_1 b_2^{\mathcal{N}} &\geq 0\ IC_1\\
(\alpha_1-\alpha_2)\,3x_1+(1-\varepsilon)\,(\alpha_1 b_2+\alpha_2 b_3)+(2+\varepsilon)\,\alpha_2 b_3^{\mathcal{N}}-(4-\varepsilon)\,\alpha_1 b_2^{\mathcal{N}} &\geq 0\ IC_1^{'}
\end{aligned}$$

where IC_3 is player 3 incentives to deviate for position 2, IC_2 the player 2's incentives for no position, IC_1 player 1's for no position and $IC_1^{'}$ player 1's for position 2. Notice that IC_3 implies both IC_2 and IC_1. Now, assume that $b_2^{\mathcal{N}}=b_3^{\mathcal{N}}=0$, we have:

$$\begin{aligned}
(1-\varepsilon)\,(\alpha_1 b_2+\alpha_2 b_3)-3\alpha_2 x_3 &\geq 0\ IC_3\\
(\alpha_1-\alpha_2)\,3x_1+(1-\varepsilon)\,(\alpha_1 b_2+\alpha_2 b_3) &\geq 0\ IC_1^{'}
\end{aligned}$$

Thus, these bids are compatible if the following is true:

$$\begin{aligned}
(1-\varepsilon)\,(b_2+\theta b_3)-3\theta x_3 &\geq 0\\
(1-\theta)\,3x_1+(1-\varepsilon)\,(b_2+\theta b_3) &\geq 0
\end{aligned}$$

that is if:

$$\begin{aligned}
\theta &\leq \frac{(1-\varepsilon)\,b_2}{3x_3-(1-\varepsilon)\,b_3}=\eta\\
\theta &\leq \frac{3x_1+(1-\varepsilon)\,b_2}{3x_3-(1-\varepsilon)\,b_3}
\end{aligned}$$

where the first inequality implies the second one. Thus, $b_2^{\mathcal{N}}=b_3^{\mathcal{N}}=0$ are compatible bids if $\theta\leq\eta$. □

Since, $b_1^{\mathcal{N}}$ deters any deviation for the top position, it suffices to consider the adjacent deviations.

Then, according to the claim, to ensure an efficient assignment we impose $\theta > \eta$ so that $b_2^{\mathcal{N}} > 0$ and maintain the bid $b_3^{\mathcal{N}} = 0$. Both constraints are,

$$\begin{aligned}(1-\varepsilon)\sum_{i=1}^{2}\alpha_i b_{i+1} - 3\alpha_2 x_3 + (3\alpha_2 - (1-\varepsilon)\alpha_1) b_2^{\mathcal{N}} &\geq 0 \; IC_3 \\ (\alpha_1 - \alpha_2) 3x_1 + (1-\varepsilon)\sum_{i=1}^{2}\alpha_i b_{i+1} - (4-\varepsilon)\alpha_1 b_2^{\mathcal{N}} &\geq 0 \; IC_1^{'}\end{aligned}$$

as $x_3 < x_1$ if player 3's incentive to move for position 2 binds, it should also be the case for player 1. Thus, we are left with the constraint IC_3 in equilibrium which implies,

$$\begin{aligned}(1-\varepsilon)\sum_{i=1}^{2}\alpha_i b_{i+1} - 3\alpha_2 x_3 + (3\alpha_2 - (1-\varepsilon)\alpha_1) b_2^{\mathcal{N}} &= 0 \\ b_2^{\mathcal{N}} &= \frac{1}{3\alpha_2 - (1-\varepsilon)\alpha_1}\left(3\alpha_2 x_3 - (1-\varepsilon)\sum_{i=1}^{2}\alpha_i b_{i+1}\right)\end{aligned}$$

Notice that, $b_2^{\mathcal{N}} < x_3$ since we can re-write the last expression as,

$$\begin{aligned}b_2^{\mathcal{N}} &= \frac{1}{3\alpha_2 - (1-\varepsilon)\alpha_1}\left(3\alpha_2 x_3 - (1-\varepsilon)\sum_{i=1}^{2}\alpha_i b_{i+1}\right) \\ &= x_3 - \left(\frac{(1-\varepsilon)}{(1-\varepsilon)\alpha_1 - 3\alpha_2}\left(\alpha_1 (x_3 - b_2) - \alpha_2 b_3\right)\right) \\ &= x_3 - \lambda\left(\alpha_2 x_3 - \alpha_2 b_3 - \alpha_2 x_3 + \alpha_1 x_3 - \alpha_1 b_2\right) \\ &= x_3 - \lambda\left(\pi_3 + (\alpha_1 - \alpha_2) x_3 - \alpha_1 b_2\right)\end{aligned}$$

and that $IC_1^{'}$ is satisfied with such bid,

$$\begin{aligned}(\alpha_1 - \alpha_2) 3x_1 + (1-\varepsilon)\sum_{i=1}^{2}\alpha_i b_{i+1} - (4-\varepsilon)\alpha_1 b_2^{\mathcal{N}} &\geq (\alpha_1 - \alpha_2) 3x_1 + (1-\varepsilon)\sum_{i=1}^{2}\alpha_i b_{i+1} - (4-\varepsilon)\alpha_1 x_3 \\ &\geq (\alpha_1 - \alpha_2) 3x_3 + (1-\varepsilon)\sum_{i=1}^{2}\alpha_i b_{i+1} - (4-\varepsilon)\alpha_1 x_3 \\ &\geq (1-\varepsilon)\sum_{i=1}^{2}\alpha_i b_{i+1} - (3\alpha_2 - (1-\varepsilon)\alpha_1) x_3 \geq 0\end{aligned}$$

Now, consider the case in which $b_2^{\mathcal{N}} > 0$ and $b_3^{\mathcal{N}} > 0$. This case is possible only towards the parameter ε. Assume that there exists a threshold δ so that if $\varepsilon > \delta$ then $b_2^{\mathcal{N}} > 0$ and $b_3^{\mathcal{N}} > 0$.

Both constraints equals,

$$
\begin{aligned}
(1-\varepsilon)\sum_{i=1}^{2}\alpha_i b_{i+1} - 3\alpha_2 x_3 - (1-\varepsilon)\,\alpha_2 b_3^{\mathcal{N}} + (3\alpha_2 - (1-\varepsilon)\,\alpha_1)\, b_2^{\mathcal{N}} &\geq 0 \; IC_3 \\
(\alpha_1-\alpha_2)\,3x_1 + (1-\varepsilon)\sum_{i=1}^{2}\alpha_i b_{i+1} + (2+\varepsilon)\,\alpha_2 b_3^{\mathcal{N}} - (4-\varepsilon)\,\alpha_1 b_2^{\mathcal{N}} &\geq 0 \; IC_1^{'}
\end{aligned}
$$

Both constraints will be binding at the optimum, otherwise the *intermediary firm* can decrease $b_2^{\mathcal{N}}$ and $b_3^{\mathcal{N}}$ accordingly. We get,

$$
\begin{aligned}
b_2^{\mathcal{N}} &= \frac{1}{3\alpha_2-(1-\varepsilon)\,\alpha_1}\left(3\alpha_2 x_3 - (1-\varepsilon)\sum_{i=1}^{2}\alpha_i b_{i+1} + (1-\varepsilon)\,\alpha_2 b_3^{\mathcal{N}}\right) \\
b_3^{\mathcal{N}} &= \frac{1}{(2+\varepsilon)\,\alpha_2}\left((4-\varepsilon)\,\alpha_1 b_2^{\mathcal{N}} - (1-\varepsilon)\sum_{i=1}^{2}p_i - (\alpha_1-\alpha_2)\,3x_1\right)
\end{aligned}
$$

Plug the expression of $b_2^{\mathcal{N}}$ in the one of $b_3^{\mathcal{N}}$ results in the following relation,

$$
b_3^{\mathcal{N}} = \frac{\left(x_1(\alpha_1-\alpha_2)\left((1-\varepsilon)\,\alpha_1-3\alpha_2\right) - (1-\varepsilon)\,(\alpha_1+\alpha_2)\sum_{i=1}^{2}\alpha_i b_{i+1} + (4-\varepsilon)\,\alpha_1\alpha_2 x_3\right)}{\alpha_2\left((2+\varepsilon)\,\alpha_2-(1-\varepsilon)\,2\alpha_1\right)}
$$

which we replace in the expression of $b_2^{\mathcal{N}}$ to get,

$$
b_2^{\mathcal{N}} = \frac{\left((2+\varepsilon)\,\alpha_2 x_3 - (1-\varepsilon)\sum_{i=1}^{2}\alpha_i b_{i+1} - (1-\varepsilon)\,(\alpha_1-\alpha_2)\,x_1\right)}{(2+\varepsilon)\,\alpha_2-(1-\varepsilon)\,2\alpha_1}
$$

which are the relations in the bid profile $\boldsymbol{\mu}_{\mathcal{N}}$ proposed in relations (13). To find the value of the threshold δ simply find the value of ε that solves,

$$
\begin{aligned}
\frac{\left((2+\varepsilon)\,\alpha_2 x_3 - (1-\varepsilon)\sum_{i=1}^{2}\alpha_i b_{i+1} - (1-\varepsilon)\,(\alpha_1-\alpha_2)\,x_1\right)}{(2+\varepsilon)\,\alpha_2-(1-\varepsilon)\,2\alpha_1} \\
-\frac{1}{3\alpha_2-(1-\varepsilon)\,\alpha_1}\left(3\alpha_2 x_3-(1-\varepsilon)\sum_{i=1}^{2}\alpha_i b_{i+1}\right) &= 0 \\
&\textit{and} \\
\frac{\left(x_1(\alpha_1-\alpha_2)\left((1-\varepsilon)\,\alpha_1-3\alpha_2\right) - (1-\varepsilon)\,(\alpha_1+\alpha_2)\sum_{i=1}^{2}\alpha_i b_{i+1} + (4-\varepsilon)\,\alpha_1\alpha_2 x_3\right)}{\alpha_2\left((2+\varepsilon)\,\alpha_2-(1-\varepsilon)\,2\alpha_1\right)} &= 0
\end{aligned}
$$

this returns,

$$
\varepsilon^* = \delta = \frac{\left(1+\frac{\alpha_2}{\alpha_1}\right)\sum_{i=1}^{2}\alpha_i b_{i+1} - \left(1-\frac{\alpha_2}{\alpha_1}\right)(\alpha_1-3\alpha_2)\,x_1 - 4\alpha_2 x_3}{\left(1+\frac{\alpha_2}{\alpha_1}\right)\sum_{i=1}^{2}\alpha_i b_{i+1} - \left(1-\frac{\alpha_2}{\alpha_1}\right)\alpha_1 x_1 - \alpha_2 x_3}
$$

The proof for the monotonicity and for the decrease in $\mathbf{b}$ is straightforward. Denotes by $\underline{b}_i^{\mathcal{N}}$ and $\bar{b}_i^{\mathcal{N}}$ the respective collusive bids for $\varepsilon \leq \delta$ and $\varepsilon > \delta$. $\forall \varepsilon \leq \delta$ the derivative of $\underline{b}_2^{\mathcal{N}}$ with respect to ε is given by:

$$\frac{\partial \left(\underline{b}_2^{\mathcal{N}}\right)}{\partial \varepsilon} = \frac{3\alpha_2 \left(\alpha_1 b_2 + \alpha_2 b_3 - \alpha_1 x_3\right)}{\left(3\alpha_2 - (1-\varepsilon)\,\alpha_1\right)^2} \geq 0 \tag{21}$$

and the derivative with respect to $R = (\alpha_1 b_2 + \alpha_2 b_3)$ is given by:

$$\frac{\partial \left(\underline{b}_2^{\mathcal{N}}\right)}{\partial R} = -\frac{1-\varepsilon}{3\alpha_2 - \alpha_1 (1-\varepsilon)}$$

which is negative if:

$$\begin{aligned} 3\alpha_2 - \alpha_1 (1-\varepsilon) &\geq 0 \\ \frac{\alpha_2}{\alpha_1} &\geq \frac{1-\varepsilon}{3} \end{aligned}$$

which is satisfied by the restriction that $\underline{b}_2^{\mathcal{N}} > 0$. Now, $\forall \varepsilon > \delta$ the derivatives are given by:

$$\begin{aligned} \frac{\partial \left(\bar{b}_2^{\mathcal{N}}\right)}{\partial \varepsilon} &= \frac{3\alpha_2 \left(\alpha_1 b_2 + \alpha_2 b_3 + (\alpha_1 - \alpha_2)\, x_1 - 2\alpha_1 x_3\right)}{\left(\alpha_2 (2+\varepsilon) - 2\alpha_1 (1-\varepsilon)\right)^2} \geq 0 \\ \frac{\partial \left(\bar{b}_3^{\mathcal{N}}\right)}{\partial \varepsilon} &= \frac{3 (\alpha_1 + \alpha_2) \left(\alpha_1 b_2 + \alpha_2 b_3 + (\alpha_1 - \alpha_2)\, x_1 - 2\alpha_1 x_3\right)}{\left(\alpha_2 (2+\varepsilon) - 2\alpha_1 (1-\varepsilon)\right)^2} \geq 0 \\ \frac{\partial \left(\bar{b}_2^{\mathcal{N}}\right)}{\partial R} &= \frac{-(1-\varepsilon)}{(2+\varepsilon)\,\alpha_2 - (1-\varepsilon)\, 2\alpha_1} \leq 0 \\ \frac{\partial \left(\bar{b}_3^{\mathcal{N}}\right)}{\partial R} &= \frac{-(1-\varepsilon)(\alpha_1 + \alpha_2)}{\alpha_2 \left((2+\varepsilon)\,\alpha_2 - (1-\varepsilon)\, 2\alpha_1\right)} \leq 0 \end{aligned}$$

in which the last two denominators are of positive signs if $\theta \geq 2\frac{(2+\varepsilon)}{(1-\varepsilon)}$.

A.3 Proof of corollary (1)

Take the limit as ε grows to 1 we see that collusive payoffs converge to the *non-cooperative* level we get:

$$\lim_{\varepsilon \mapsto +\infty} \left\{ \alpha_k \left(x_k - b_{k+1}^{\mathcal{N}}\right) - \left(\alpha_k b_{k+1} - \alpha_k b_{k+1}^{\mathcal{N}}\right) + \frac{1-\varepsilon}{n} \sum_{j=1}^{m} \alpha_j b_{j+1} - \frac{1-\varepsilon}{n} \sum_{j=1}^{m} \alpha_j b_{j+1}^{\mathcal{N}} \right\} \tag{22}$$

which leads to $\alpha_k (x_k - b_{k+1}) = \pi_k$. The objective of the centre is thus now to maximize the function $SP = \sum_{k=1}^{n} \left(\alpha_k (x_k - b_{k+1})\right)$ under the same *IC* constraints of the *non-cooperative* equilibrium given by conditions (19) which entails $\mathbf{b}^L$ as a natural equilibrium outcome. One can also set $\varepsilon = 1$ into the equilibrium collusive bidding functions defined in corollary (13) to get:

$$\begin{aligned} b_2^{\mathcal{N}} &= x_3 = b_2^L \\ b_3^{\mathcal{N}} &= x_1 - \frac{\alpha_1}{\alpha_2}(x_1 - x_3) = b_3^L \end{aligned}$$

which is the desired outcome.

A.4 Proof of proposition (2)

$\forall \varepsilon \in [0, \delta^*]$ the equilibrium strategy profile is characterized by the sets (12). Thus the centre's profit equals the following quantity:

$$\begin{aligned} \Gamma_{\mathcal{N}}^{'} &= \varepsilon \left\{ \alpha_1 \left(b_2 - b_2^{\mathcal{N}} \right) + \alpha_2 \left(b_3 - b_3^{\mathcal{N}} \right) \right\} \\ &= \varepsilon \left\{ R - \alpha_1 \left(\frac{3\alpha_2 x_3 - (1-\varepsilon)(\alpha_1 b_2 + \alpha_2 b_3)}{3\alpha_2 - (1-\varepsilon)\alpha_1} \right) \right\} \end{aligned}$$

and $\forall \varepsilon \in (\delta^*, 1)$, by the equilibrium set (13) it takes the following quantity:

$$\begin{aligned} \Gamma_{\mathcal{N}}^{''} = \varepsilon \Bigg\{ & R - \alpha_1 \left(\frac{(2+\varepsilon)\alpha_2 x_3 - (1-\varepsilon)(\alpha_1 b_2 + \alpha_2 b_3) - (1-\varepsilon)(\alpha_1 - \alpha_2) x_1}{(2+\varepsilon)\alpha_2 - (1-\varepsilon) 2\alpha_1} \right) \\ & -\alpha_2 \left(\frac{x_1(\alpha_1 - \alpha_2)((1-\varepsilon)\alpha_1 - 3\alpha_2) - (1-\varepsilon)(\alpha_1 + \alpha_2)(\alpha_1 b_2 + \alpha_2 b_3) + (4-\varepsilon)\alpha_1 \alpha_2 x_3}{\alpha_2((2+\varepsilon)\alpha_2 - (1-\varepsilon)2\alpha_1)} \right) \Bigg\} \end{aligned}$$

The first and second derivative of $\Gamma_{\mathcal{N}}^{'}$ are respectively equal to:

$$\frac{\partial}{\partial \varepsilon}\left(\Gamma_{\mathcal{N}}^{'}\right) = 3\alpha_2 (3\alpha_2 - \alpha_1) \frac{(\alpha_1 b_2 + \alpha_2 b_3 - \alpha_1 x_3)}{(3\alpha_2 - (1-\varepsilon)\alpha_1)^2}$$

which is positive if $3\alpha_2 \geq \alpha_1$ and

$$\frac{\partial^2}{\partial \varepsilon^2}\left(\Gamma_{\mathcal{N}}^{'}\right) = -6\alpha_1\alpha_2 (3\alpha_2 - \alpha_1) \frac{(\alpha_1 b_2 + \alpha_2 b_3 - \alpha_1 x_3)}{(3\alpha_2 - (1-\varepsilon)\alpha_1)^3}$$

which is of negative sign whenever $3\alpha_2 \geq \alpha_1$. Thus on the domain $[0, \delta^*)$ the intermediary firm profit function is concave. The function is non-differentiable in $\varepsilon = \delta^*$ and the first and second derivative of $\Gamma_{\mathcal{N}}^{''}$ are respectively given by:

$$\frac{\partial}{\partial \varepsilon}\left(\Gamma_{\mathcal{N}}^{''}\right) = -6\alpha_2 (\alpha_1 - \alpha_2) \frac{(\alpha_1 b_2 + \alpha_2 b_3 + x_1(\alpha_1 - \alpha_2) - 2\alpha_1 x_3)}{((2+\varepsilon)\alpha_2 - (1-\varepsilon)2\alpha_1)^2}$$

which is of negative sign and

$$\frac{\partial^2}{\partial \varepsilon^2}\left(\Gamma_{\mathcal{N}}^{''}\right) = 12\alpha_2 (\alpha_1 - \alpha_2)(2\alpha_1 + \alpha_2) \frac{(\alpha_1 b_2 + \alpha_2 b_3 + x_1(\alpha_1 - \alpha_2) - 2\alpha_1 x_3)}{((2+\varepsilon)\alpha_2 - (1-\varepsilon)2\alpha_1)^2}$$

which is of positive sign. Thus, the profit function is a convex function over the domain $(\delta^*, 1)$.

A.5 Proof of claim (1)

From proposition (2), we know that the *intermediary firm* as no incentives to set $\varepsilon > \delta$, thus, it suffices to focus on the bid functions of relation (12). Take the equilibrium bid profile $\boldsymbol{\mu}_{\mathcal{N}}$ of proposition (1) and rearrange the expressions so that bids become a function of θ. From proposition (2), we know that the *intermediary firm* as no incentives to set $\varepsilon > \delta$, thus, it suffices to focus on the bid functions of relation (12).

Consider the situation in which $\alpha_1 \mapsto \infty$ so that $\theta = 0$, from claim (2) we have that $b_2^{\mathcal{N}} = b_3^{\mathcal{N}} = 0$. Hence, the auctioneer's revenues would drop to zero. If now, the auctioneer set $\alpha_1 = \alpha_2$ so that $\theta = 1$. Rearrange relation (12) to get,

$$b_2^{\mathcal{N}}(\theta) = \frac{3\theta x_3 - (1-\varepsilon)(b_2 + \theta b_3)}{3\theta - (1-\varepsilon)}$$

from proposition it is optimal for the *intermediary firm* to set $\varepsilon = \delta$, thus we get,

$$b_2^{\mathcal{N}} = \frac{\alpha_2 x_3 + \alpha_2 (b_2 - b_3) - 2\alpha_1 b_2}{2\alpha_2 (x_3 - \alpha_1 b_2 + \alpha_2 b_3)}$$

which equals $\frac{1}{2}x_3$ if the outside bid profile is assumed to be $\mathbf{b}^l = \left(b_2^l = x_3, b_3^l = x_1 - \frac{\alpha_1}{\alpha_2}(x_1 - x_3)\right)$ and x_3 if $\mathbf{b}^l = \left(b_2^l = x_3, b_3^l = 0\right)$ is considered. As a result, the auctioneer's revenue cannot be higher than $\mathbf{b}^l$.

A.6 Proof of proposition (4)

In order for player two to be indifferent between winning the second position at price $b_3^{\mathcal{N}}$ and winning the top position at price $b_2^{\mathcal{N}}$, the following relation should be satisfied:

$$\alpha_2 \left(x_2 - b_3^{\mathcal{N}}\right) - \omega_2 + \frac{1-\varepsilon}{3}\Pi_{\mathcal{N}} = \alpha_1 \left(x_2 - b_2^{\mathcal{N}}\right) - \omega_2$$

For player three to be indifferent between being assigned to the second position and not being assigned under the collusive agreement it should be:

$$\frac{1-\varepsilon}{3}\Pi_{\mathcal{N}} = \alpha_2 \left(x_3 - b_3^{\mathcal{N}}\right) \tag{23}$$

Rearranging both relations, we obtain the following pair of equations:

$$\alpha_1 b_2^{\mathcal{N}} = \frac{3}{2+\varepsilon}\left(x_2(\alpha_1 - \alpha_2)\right) + \frac{4-\varepsilon}{2+\varepsilon}\alpha_2 b_3^{\mathcal{N}} - \frac{1-\varepsilon}{2+\varepsilon}\left(\sum_{i=1}^{2} P_i^v\right) \tag{24}$$

$$\alpha_2 b_3^{\mathcal{N}} = \frac{3}{2+\varepsilon}\alpha_2 x_3 + \frac{1-\varepsilon}{2+\varepsilon}\left(\alpha_1 b_2^{\mathcal{N}} - \sum_{i=1}^{2} P_i^v\right) \tag{25}$$

Recall that $P_i^v = \sum_{k=i+1}^{m+1} x_k(\alpha_{k-1} - \alpha_k)$. Rearranging terms, plugging (25) into (24) and using equation (11), the following relation is obtained:

$$\begin{aligned}\alpha_1 b_2^{\mathcal{N}} &= x_2(\alpha_1 - \alpha_2) + \alpha_2 x_3 = P_1^v \\ \alpha_2 b_3^{\mathcal{N}} &= \alpha_2 x_3 = P_2^v\end{aligned} \tag{26}$$

The solution corresponds for any $\varepsilon \in [0,1]$ to the same equilibrium bids and payments of the *VCG*-equivalent equilibrium bids profile of equation (11) and thus results in the same outcome as the *symmetric non-cooperative* one.

B Figures and Tables

We run 1000 instance of the one-shot *GSP* game in which following Cary et al. (2008), each valuation is drawn from a distribution $G(x) \sim \mathcal{N}(500, 200)$ setting the value of X_1, X_2 and X_3 respectively to $x_1 = 592.7$, $x_2 = 565.535$ and $x_3 = 437.331$. On average, we observe a *ctr* of 0.23% on higher positions which allows us to reasonably set $\lambda = 0.23$ giving $E(\alpha) = 4.3$ (Synodiance 2013 synodiance.ctr.study2013) thus $H(\alpha) \sim \mathcal{E}_{(0.23)}$. The exponential law will generate numbers lying between 0 and 1 which can be straightly interpreted as clicks probability or clicks rates.

B.1 Bids, profits and revenues

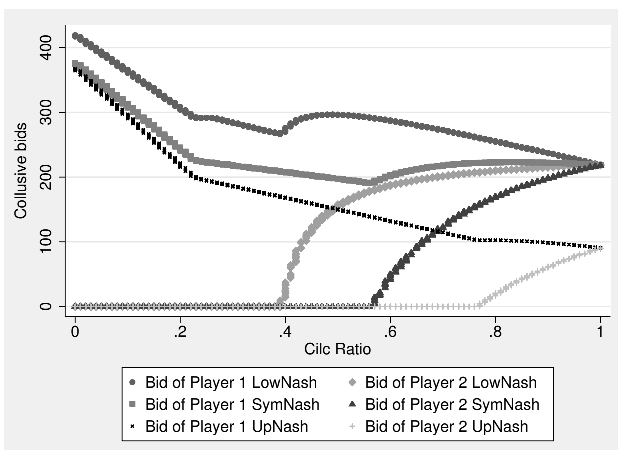
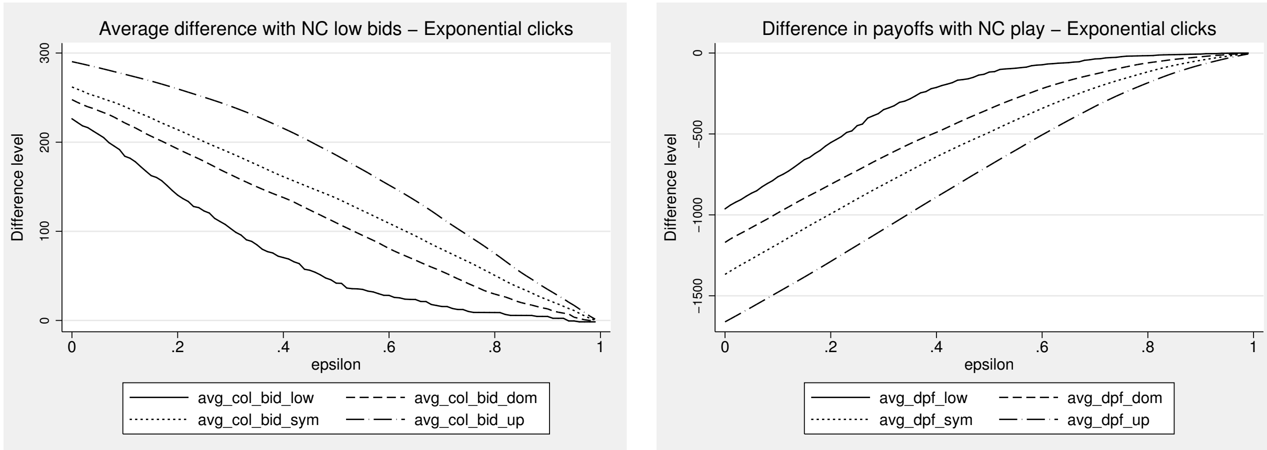


Figure 5: Equilibrium collusive bids when $\varepsilon = 0$ against θ respectively with outside bid profile $\mathbf{b}^l, \mathbf{b}^v, \mathbf{b}^u$ when $H(\alpha) \sim \mathcal{E}_{(0,23)}$.



(a) Difference in bids

(b) Difference in payoffs

Figure 6: Average difference with the bids sustaining $\mathbf{b}^l$ and payoffs against each $\mathbf{b}$ when the centre set a fee $\varepsilon \geq 0$.

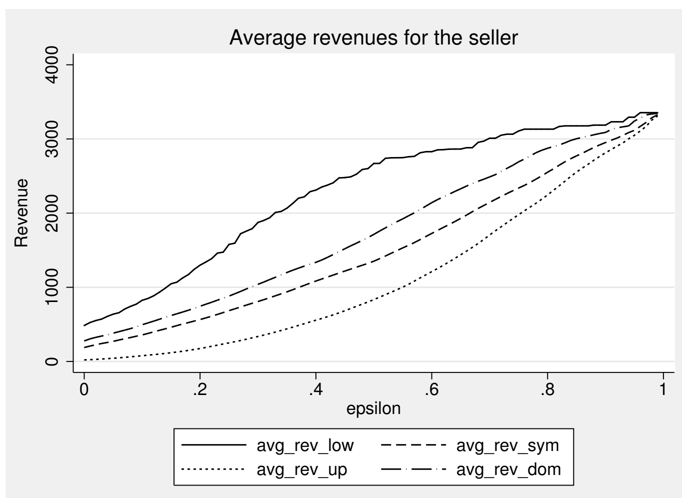


Figure 7: Average seller's revenue according to $\mathbf{b}^l, \mathbf{b}^d, \mathbf{b}^v$ and $\mathbf{b}^u$ when the centre set a fee $\varepsilon \geq 0$.

B.2 Comparison between *non-cooperative* outcome and collusive outcome and thresholds for ε

	Benevolent Centre									
	Non-Cooperative Play					Cartel				
	$\mathbf{b}^l$	$\mathbf{b}^d$	$\mathbf{b}^v$	$\mathbf{b}u$	Average	$\mathbf{b}^l$	$\mathbf{b}^d$	$\mathbf{b}^v$	$\mathbf{b}^u$	Average
Surplus										
Player 1	1024.32	1024.32	466.81	60.98	644.11	2039.84	2231.134	1878.852	1756.32	1976.53
Player 2	700.97	287.65	287.65	0	319.06	1716	1494.49	1699.67	1695.34	1651.52
Player 3	.	.	.	.	.	1015.54	1206.84	1412.02	1695.34	1332.43
Contributions										
Player 1	.	.	.	.	.	2478.76	2639.27	3254.81	3817.13	3047.5
Player 2	.	.	.	.	.	567.86	981.24	981.24	1268.9	949.8
Player 3	.	.	.	.	.	0	0	0	0	
Average Contributions	.	.	.	.	.	1015.542	1206.84	1412.02	1332.43	
Seller Revenue	3450.82	3864.2	4421.7	5115.17	4212.9	404.2	243.7	185.64	29.15	215.67
Total Surplus	5176.15	5176.15	5176.15	5176.15	5176.15	5176.15	5176.15	5176.15	5176.15	5176.15
Cartel Surplus	.	.	.	.	.	4771.95	4932.46	4990.5	5147	4960.48
Bids										
Player 1	453.53	514.83	592.71	592.71	538.44	304.38	274.31	244.81	194.88	254.6
Player 2	437.3	437.3	514.83	581.96	492.86	78.89	49.9	38.98	7.04	43.73
Player 3	170.14	437.3	437.3	565.54	402.59	0	0	0	0	0
Average Bids	353.67	463.16	514.96	580.07		127.76	108.1	94.6	67.31	

Table 1: Equilibrium outcomes with a benevolent centre for $H(\alpha) \sim \mathcal{E}_{(0.23)}$ and $G(x) \sim \mathcal{N}_{(500,200)}$

	Corresponding NC outcomes				
	$\mathbf{b}^l$	$\mathbf{b}^d$	$\mathbf{b}^v$	$\mathbf{b}^u$	Average
Surplus					
Player 1	790.28	790.28	633.53	104.9	579.77
Player 2	685.22	495.27	495.27	0	418.94
Player 3	.	.	.	.	.
Seller Revenue	3723.88	3913.85	4070.6	5094.43	4200.69
Total Surplus	5199.3	5199.3	5199.3	5199.3	5199.3
Bids					
Player 1	431.12	466.65	592.71	592.71	520.79
Player 2	437.3	437.3	466.65	571.75	478.25
Player 3	385.58	437.3	437.3	565.54	456.45
Average Bids	418	447.1	498.9	576.67	

Table 3: Equilibrium *NC* outcomes corresponding to the non-neutral centre case for $H(\alpha) \sim \mathcal{E}_{(0.23)}$ and $G(x) \sim \mathcal{N}_{(500,200)}$

	Active Centre														
	ε'					$\varepsilon = \delta^*$					ε''				
	$\mathbf{b}^l$	$\mathbf{b}^d$	$\mathbf{b}^v$	$\mathbf{b}^u$	Mean	$\mathbf{b}^l$	$\mathbf{b}^d$	$\mathbf{b}^v$	$\mathbf{b}^u$	Mean	$\mathbf{b}^l$	$\mathbf{b}^d$	$\mathbf{b}^v$	$\mathbf{b}^u$	Mean
Threshold	0.085	0.115	0.136	0.236	.	0.169	0.23	0.271	0.472	.	0.254	0.345	0.407	0.709	.
Surplus															
Player 1	1554.94	1604.94	1487.98	1203.46	1462.83	1448.38	1448.38	1291.63	763.1	1237.87	790.272	984.68	867.175	324.98	741.77
Player 2	1449.94	1309.97	1349.76	1098.47	1302.03	1343.4	1153.41	1153.41	658.10	1077.07	685.28	689.71	728.85	219.98	580.98
Player 3	764.66	814.67	854.46	1098.47	883.06	658.1	658.1	658.1	658.1	658.1	0	194.41	233.65	219.98	162.01
Centre Payoff	201.31	314.73	410.47	1025.63	488.03	366.8	556.77	713.52	1737.35	843.61	0	287.91	458.4	1455.03	550.33
Seller Revenue	1228.58	1155.12	1096.76	773.41	1063.47	1382.78	1382.78	1382.78	1382.78	1382.78	3723.9	3042.72	2911.25	2979.45	3164.33
Total Surplus	5199.43	5199.43	5199.43	5199.43	5199.43	5199.43	5199.43	5199.43	5199.43	5199.43	5199.43	5199.43	5199.43	5199.43	5199.43
Cartel Surplus	3769.54	3729.58	3692.2	3400.4	3647.93	3449.86	3259.88	3103.14	2079.31	2973.05	1475.55	1868.8	1829.78	764.95	1484.8
Contributions															
Player 1	995.7	1069.15	1284.28	2136.15	1371.34	841.5	841.5	998.25	1526.78	1052	0	243.38	446.24	947.5	409.23
Player 2	1499.6	1689.58	1689.58	2184.9	1765.9	1499.6	1689.58	1689.58	2184.9	1765.9	0	627.76	713.1	1167.49	627.1
Player 3	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
Average	831.77	919.58	991.28	1440.34	.	780.37	843.69	895.94	1237.22	.	0	290.38	386.45	705	.
Bids															
Player 1	283.71	274.37	266.93	218.38	260.85	303.73	303.73	303.73	303.73	303.73	437.3	397.39	389.06	390.81	403.65
Player 2	240.38	226.73	215.9	152.52	208.88	269.17	269.17	269.17	269.17	269.17	437.3	388.1	378.43	382.65	396.63
Player 3	.	.	.	.	.	.	.	.	.	.	385.58	274.14	253.07	265.43	294.56
Average Bids	174.7	167.03	160.94	123.63	.	190.97	190.97	190.97	190.97	190.97	420.1	353.21	340.18	346.3	.

Table 2: Collusive outcomes when the centre set its share to $\varepsilon = \{0.5\delta^*, \delta^*, 1.5\delta^*\}$ for $H(\alpha) \sim \mathcal{E}_{(0.23)}$ and $G(x) \sim \mathcal{N}_{(500,200)}$

	Threshold δ^*			
	b^l	b^d	b^{ud}	b^u
δ^*	0.143	0.241	0.318	0.516

Table 4: Threshold δ^* values according to $\mathbf{b}^l, \mathbf{b}^d, \mathbf{b}^{ud}, \mathbf{b}^u$

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